# String effects in Lattice Gauge Theories

Bari, September 2002

Based on:

M.C., M.Panero and P. Provero
"String effects in Polyakov loop correlators"
JHEP 0206 (2002) 061

M.C., M. Hasenbusch, M.Panero and P. Provero
"String effects in the 3d gauge Ising model"
Preprint DFTT

M.C., M. Pepe and A. Rago
"String universality in LGT."
in preparation

# Plan of the Talk

- A brief history of the effective string picture of confinement

- String corrections in Wilson loop expectation values.

- String corrections in Polyakov loop correlators.

- Higher order corrections: the Nambu-Goto string.

- Comparison with MC data.

- Conclusions.

# A brief history of effective string picture of confinement

**1974-1979** Nielsen-Olesen, 't Hooft, Wilson, Polyakov, Nambu....

Conjecture:

Two color sources in a confining gauge theory are bound together by a thin flux tube, which can fluctuate like a massless string.

Problems:

- critical dimensions $d_c \neq 4$
- string spectrum versus meson spectrum

Seminal ideas:

- large N limit
- use of duality to understand confinement
- 3d gauge Ising model described by a "fermionic string" (Polyakov)

**1980** Lüscher, Symanzik, Weisz, Münster..

Universal signatures of the quantum fluctuations of the confining string:

- Universal quantum corrections ("Lüscher term") to the potential.

$$V(R) \sim \sigma R - \frac{c\pi}{24R}$$

$c = d - 2$ for a bosonic string.

- Logarithmic increase of the flux tube thickness.

These effects could in principle be observed in LGT

**1982-1986** Alvarez, Pisarski, Olesen ....

Finite T version of universal string corrections

- Deconfinement driven by the quantum fluctuations of the string.
- Effective string prediction for the ratio $T_c/\sqrt{\sigma}$.

**1985** Olesen, Alvarez ....

The flux tube as an **effective string**

- It can only describe the "large distance" behaviour of the interquark potential
- Complete independence from the "fundamental" (super)string.

**1983-1987** Dietz and Filk + Nordita group + Torino group + de Forcrand, et al.

Functional form of the effective string corrections

- Extension of the universal quantum corrections to other geometries and to finite size shapes.
- Higher order (non-universal) corrections. (Nambu-Goto string)
- Connection with CFT's

**1980-1990**

First attempts to test effective string predictions with MC simulations

- Main problem: too large statistical and systematic errors.
- Results: qualitative agreement with bosonic effective string predictions, but no clearcut evidence. In some cases MC simulations are compatible with $c = 0$

**1996-2001** Torino group + M. Hasenbusch and K.Pinn

Using duality high performance algorithms can be constructed for 3d gauge Ising model:

- (1996) test of the log increase of the flux tube width with the distance.
- (1997) test of the effective string predictions for Wilson loops
- (2001) test of the effective string predictions for Polyakov loop correlators

Results:

- Good agreement with the bosonic string predictions in all cases
- The agreement remains good even at very short scales
- In some cases it turns out that higher order effects are not negligible. In all these cases they seem to be accurately modeled by assuming a Nambu-Goto type effective string action and computing its partition function at next-to-leading order.

**1998** Witten, Maldacena .....

AdS/CFT correspondence:

- A "fundamental" string description for large N d=4 supersymmetric QCD is reached
- It is possible that a similar description also holds for ordinary QCD.
- The AdS string is completely different from the effective string. It has no relation with the color flux tube.

**1999-2001** Lucini and Teper, Necco and Sommer,...

First unambiguous evidences (with standard MC algorithms) in favour of a bosonic effective string theory in non-abelian LGT's.

**2001-2002** Lüscher and Weisz.

A new "multilevel" algorithm is proposed for non-abelian LGT's.

- Exponential reduction of statistical errors: large interquark distances can be reached
- $SU(3)$ in d=3 is perfectly described by a pure bosonic effective string theory
- $SU(3)$ in d=4 is well described by a pure bosonic effective string theory but higher order corrections are present.
- Unexpected: the transition from perturbative to string behaviour takes place at surprisingly small distances.

# Wilson loops

Wilson loops are classically expected to obey the famous area-perimeter-constant law:

$$< W(R, L) >= e^{-(\sigma RL + p(R+L) + k)}$$

This law is indeed very well verified in the strong coupling regime (before the roughening transition).

However in the rough phase it must be modified. One must multiply it by the partition function of the $2d$ QFT describing the quantum fluctuations of the flux tube.

$$< W(R, L) >= e^{-(\sigma RL + p(R+L) + k)} Z_q(R, L)$$

For the free bosonic string $Z_q(R, L)$ is simply the partition function of $d - 2$ free massless scalar fields living on the rectangle defined by the Wilson loop: $R \times T$

This partition function is given by a divergent, infinite product of eigenvalues which must be regularized. This regularization can be performed in various different ways ($\zeta$ function,momentum space cutoff, lattice cutoff....) leading to the following result:

$$Z_q(R,L) \propto \left[\frac{\eta(\tau)}{\sqrt{R}}\right]^{-\frac{d-2}{2}}$$

where $\eta(\tau)$ is the Dedekind $\eta$ function and $\tau = iL/R$.

The $L \leftrightarrow R$ simmetry is ensured by this identity

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\;\eta(\tau)$$

known as the "modular" transformation of the $\eta$.

Defining the free energy as

$$F(R,L) \equiv -\log < W(R,L) >$$

and expanding the Dedekind function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad ; \quad q = e^{2\pi i \tau} \quad ,$$

(This expansion holds only if $i\tau < -1$, i.e. $L > R$) one finds

$$F(R, L) = \sigma RL + p(R + L) + k$$

$$-(d-2)\left[\frac{\pi L}{24R} + \frac{1}{4}\log R\right] + \ldots$$

The following ratio is particularly useful to single out the effective string contribution from a collection of Wilson loops (it requires a very precise knowledge of $\sigma$):

$$R(L,n) \equiv \frac{\langle W(L+n, L-n)\rangle}{\langle W(L,L)\rangle} \exp(-n^2\sigma)$$

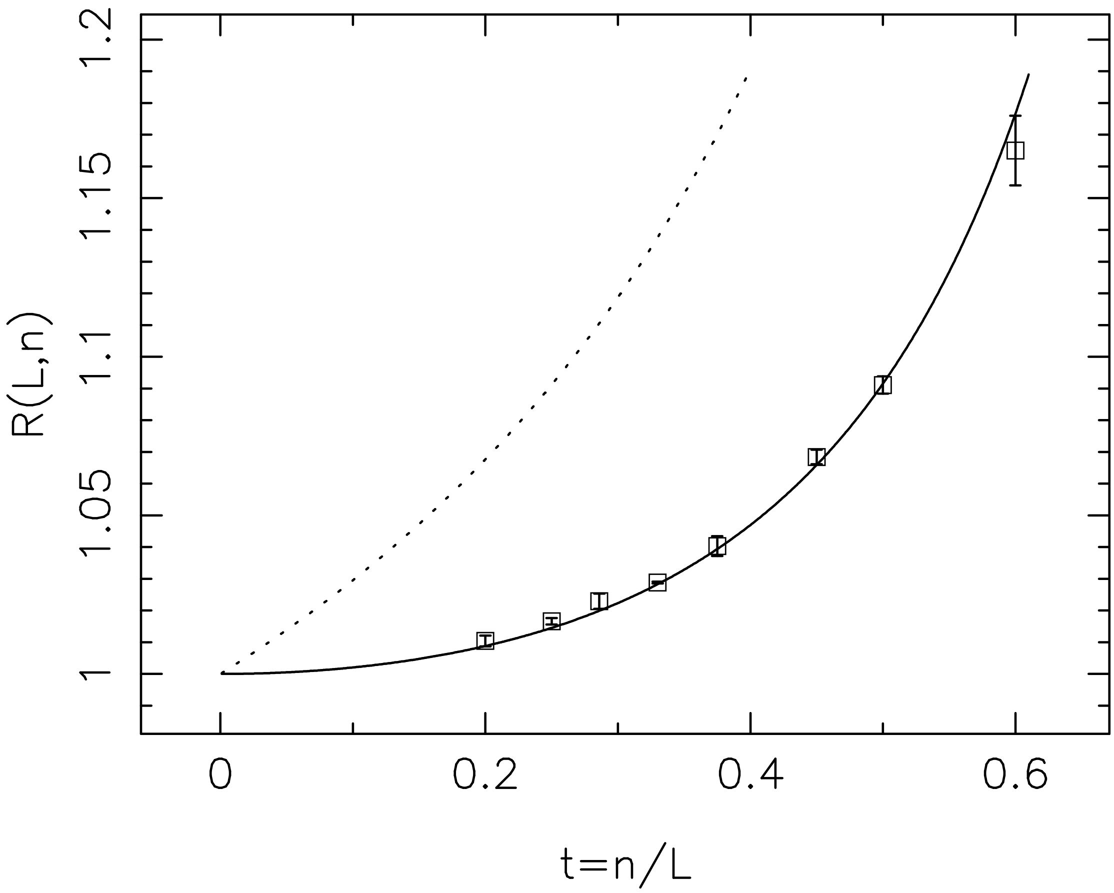
It is easy to see that $R(L,n)$ depends only on $t = n/L$:

$$R(L,n) = F(t) = \left[\frac{\eta(i)\sqrt{1-t}}{\eta\left(i\frac{1+t}{1-t}\right)}\right]^{1/2}$$

and does not contain any adjustable parameters.

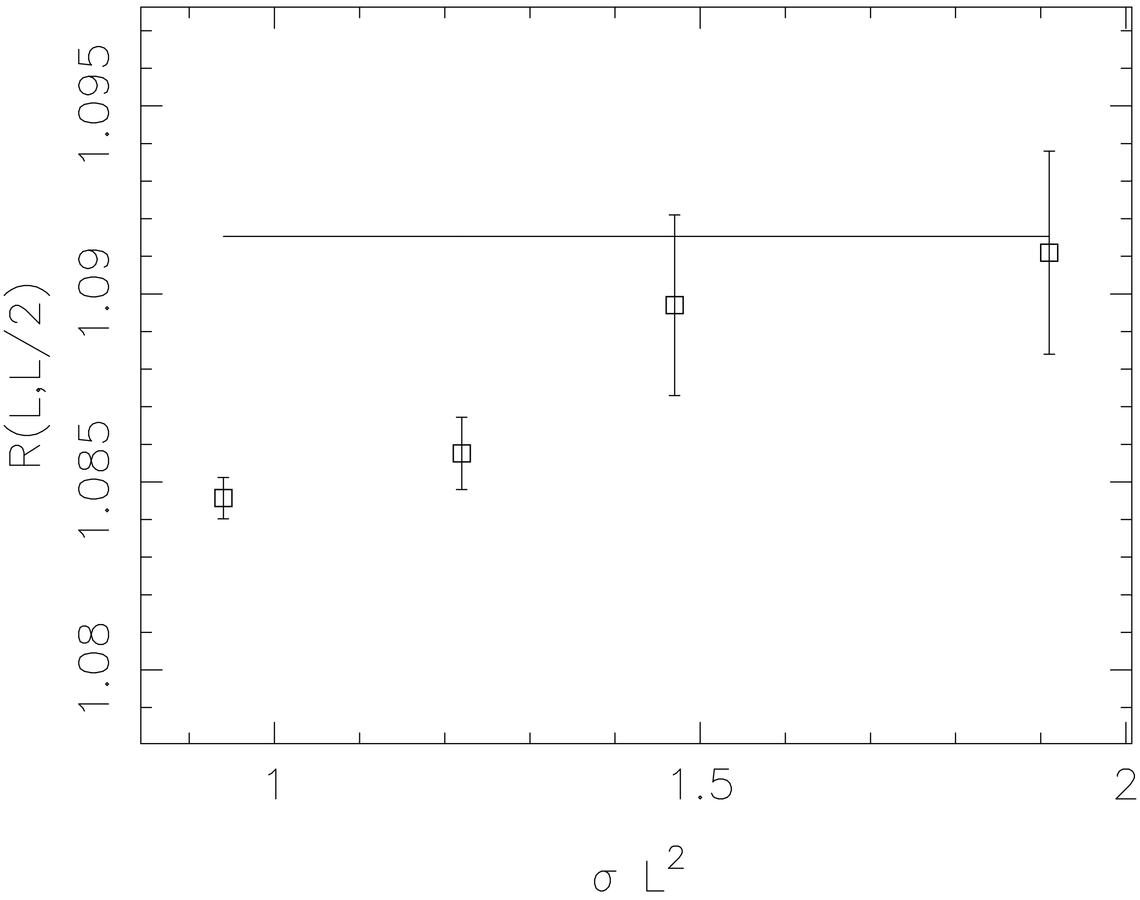
This behaviour was succesfully tested in the 3d Ising gauge case in

*M.C. et al Nucl. Phys. B* ***486*** *(1997) 245*



## Range of validity.

By varying $L$ we can test the range of validity of the free bosonic effective string.




*Finite–size effects for small Wilson loop. The prediction of the free string model is* $R(L, L/2) = 1.09153\ldots$ *(straight line).*

It turns out that L must be such that $\sigma L^2 > 1.5$. The minimum value of the interquark distance, below which the effective string picture breaks down is the flux tube thickness $R_c$.

Independent observations, based on the study of high T spacelike wilson loops suggest that $R_c \sim 1/T_c$. which gives $\sigma R_c^2 \sim 1$. The deviations that we observe in the range

$$1 < \sigma L^2 < 1.5$$

are most probably due to the self-interaction terms in the string action.

Similar results are also found in d=4 SU(3) LGT. The following figures are taken from the paper:

*S. Necco and R. Sommer, Nucl.Phys. B622 (2002) 328*

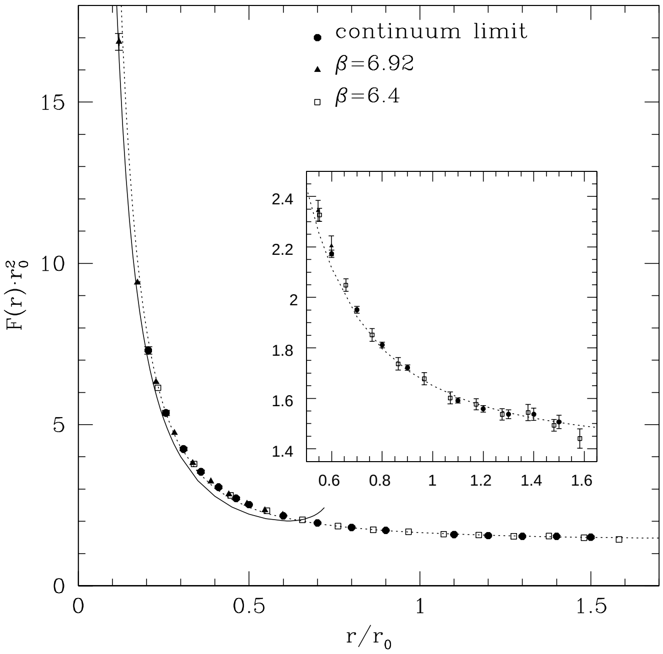
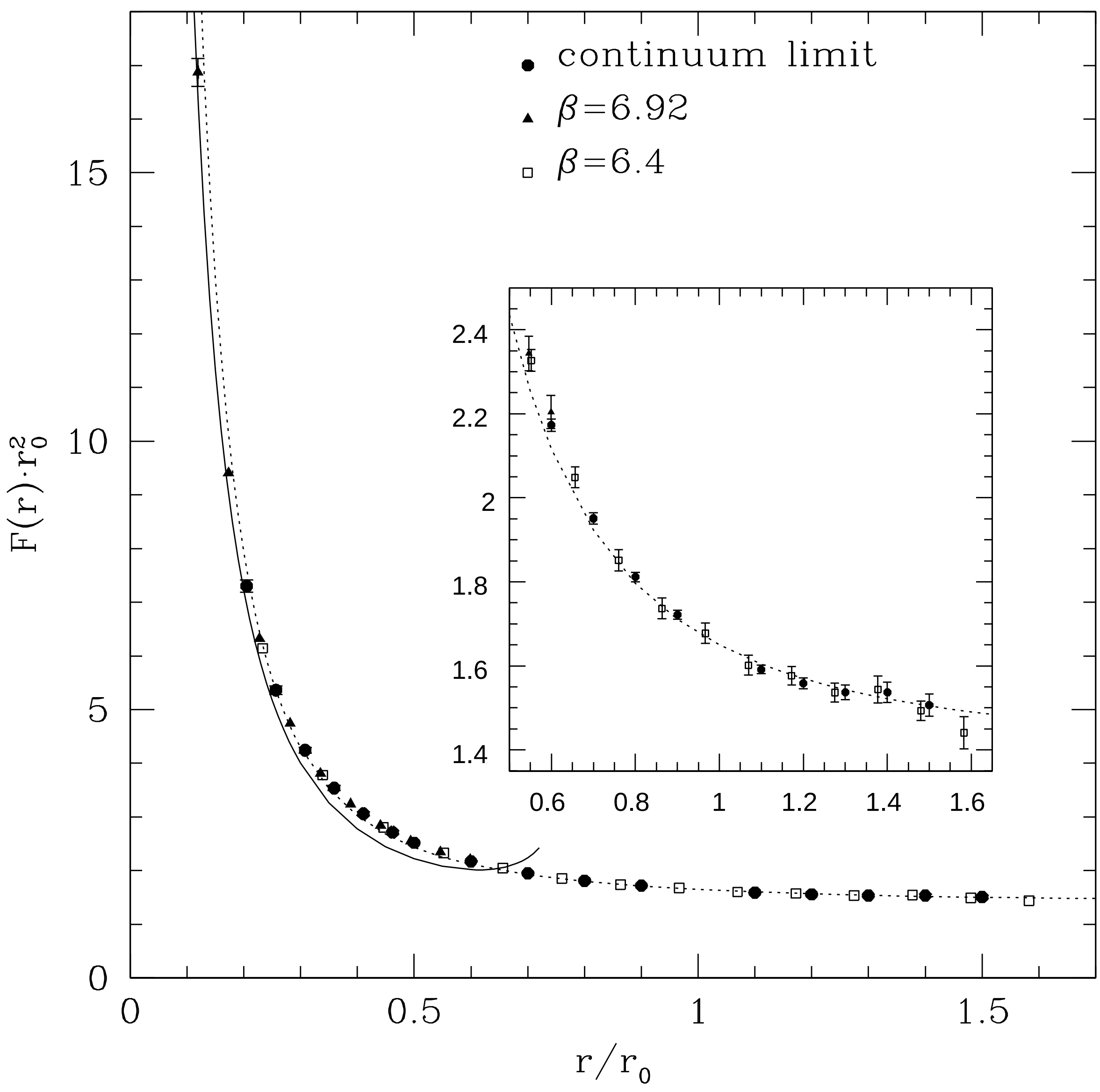



Figure 1: The force in the continuum limit and for finite resolution, where the discretization errors are estimated to be smaller than the statistical errors. The full line is the perturbative prediction with $\Lambda_{\overline{\mathrm{MS}}}\, r_0 = 0.602$. The dashed curve corresponds to the bosonic string model normalized by $r_0^2 F(r_0) = 1.65$.


continuum limit
β=6.92
β=6.4
F(r)·r₀²
r/r₀
0
5
10
15
0.5
1
1.5
2.4
2.2
2
1.8
1.6
1.4
0.6
0.8
1.2

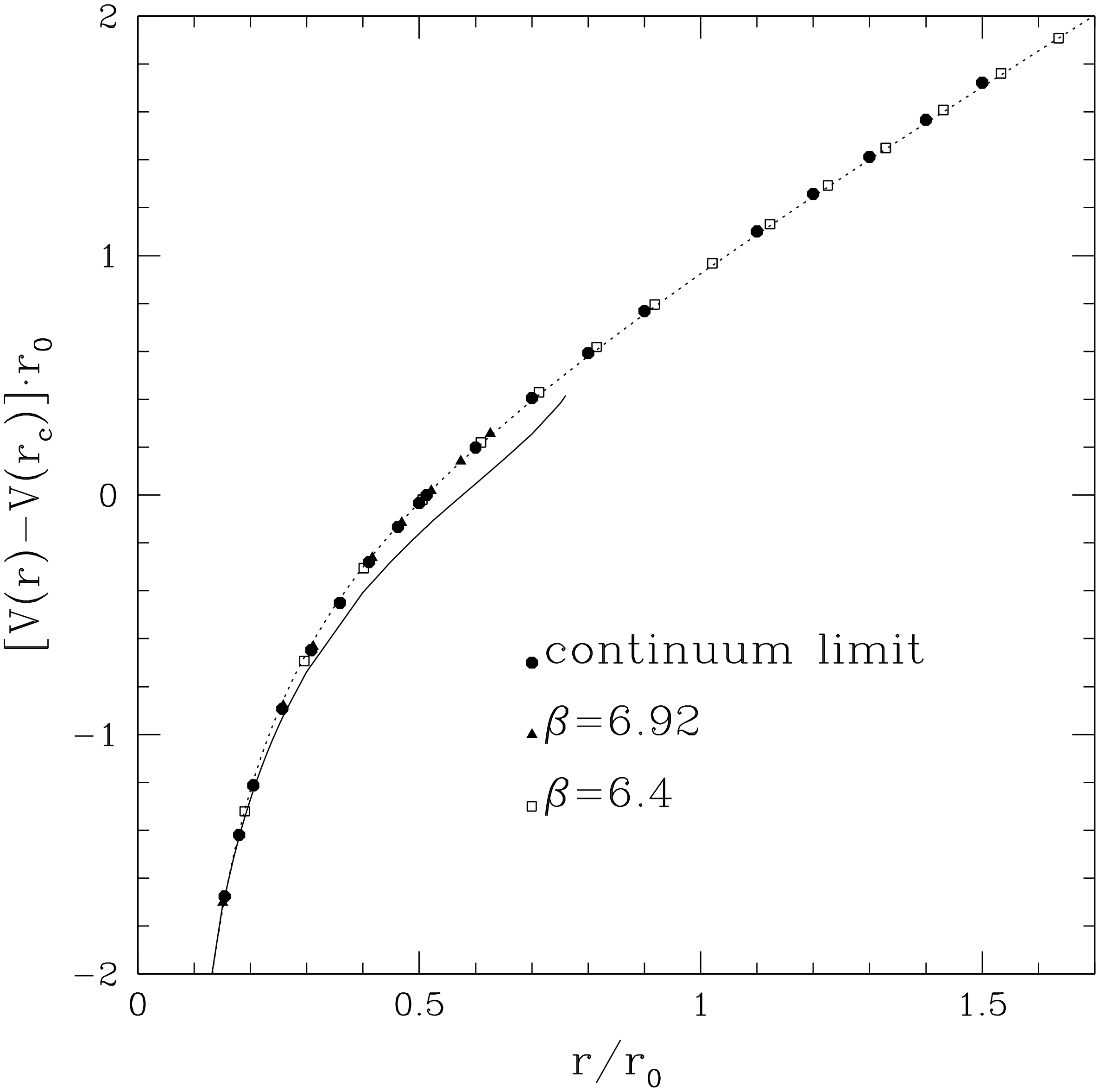



Figure 2: The static potential. The dashed line represents the bosonic string model and the solid line the prediction of perturbation theory as detailed in the text.

# Polyakov loop correlators

The peculiar geometry of the Polyakov loop correlators implies that they are perfect tools to explore the range of scales where deviations with respect to the free bosonic effective string appear.

Important observation:

In the Wilson loop geometry $(T = 0)$ $RL$ is simply the area of the loop. One can always choose large enough Wilson loops so as to reach the free string limit.

In the finite temperature geometry $L = 1/T$. The free string limit is reached only for very low temperatures. In particular at intermediate temperatures (say, $T \geq T_c/3$) higher order effects (which encode the self-interaction of the bosonic fields) become important and cannot be neglected.

Let us define the free energy as

$$G(R) = \langle P(0) P^\dagger(R) \rangle = \exp\left[-F(R,L)\right]$$

$F(R,L)$ depends on the inverse temperature $L \equiv 1/T$ ($i.e.$ the lattice size in the compactified time direction) and the distance $R$, and is given by a classical and a quantum contribution:

$$F(R,L) = F_{\rm cl}(R,L) + F_{\rm q}(R,L)$$

The classical term corresponds to the area law:

$$F_{\rm cl}(R,L) = \sigma_0 L R + k(L)$$

while the quantum term turns out to be:

$$F_{\mathrm{q}}(R, L) = (d-2)\log\eta(\tau) \qquad \tau \equiv \frac{iL}{2R}$$

where $\eta$ is again the Dedekind function

*M. Minami, Prog. Theor. Phys.* **59** *(1978) 1709.*

*P. de Forcrand, G. Schierholz, H. Schneider and M. Teper, Phys. Lett. B* **160** *(1985) 137.*

*M. Flensburg and C. Peterson, Nucl. Phys. B* **283** *(1987) 141.*

Important observation: Due to the modular transformation

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\;\eta(\tau)$$

the asymptotic expansion is different in the two regimes:

$2R < L$

$$F_q(R,L) = (d-2)\left[-\frac{\pi L}{24R} + \sum_{n=1}^{\infty}\log(1-e^{-\pi nL/R})\right]$$

$2R > L$

$$F_q(R,L) = (d-2)\left[-\frac{\pi R}{6L} + \frac{1}{2}\log\frac{2R}{L}\right]$$

$$+(d-2)\left[\sum_{n=1}^{\infty}\log(1-e^{-4\pi nR/L})\right]$$

Comments:

- For $R > L/2$ the string correction is linear in R and acts as a finite temperature renormalization of the string tension:

$$\sigma(T) = \sigma_0 - (d-2)\frac{\pi T^2}{6}$$

- For $R \sim L$ the log term cancels the linear one and the string correction vanishes. This explains why it is so difficult to observe string corrections in the Polyakov geometry

# String self-interaction terms.

Simplest option: Nambu-Goto string.

$$A[\phi] = \int_0^{L_1} dx \int_0^{L_2} dy \sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} \; .$$

where $\phi$ denotes the transverse displacement of the surface with respect to the plane joining the two Polyakov loops. There are at least two arguments in favour of this choice:

- Within the framework of the Nambu Goto action one obtains (*R. D. Pisarski and O. Alvarez, Phys. Rev. D* **26** *(1982) 3735. P. Olesen, Phys. Lett. B* **160** *(1985) 408.*)

  $$T_c^2 = \frac{3\sigma_0}{(d-2)\pi}$$

  which turns out to be in good agreement with MC results for various LGT both in d=3 and d=4.

- In the dual problem of the interface behaviour in the 3d Ising spin model it correctly describes higher order (short range) corrections in the interface free energy (*M.C et al. Nucl. Phys. B* **432** *(1994) 590*)

The next to leading order in $1/\sigma$ of the free energy can be evaluated in the framework of the zeta function regularization (*K. Dietz and T. Filk, Phys. Rev. D* **27** *(1983) 2944.*) the result in $d = 3$ is:

$$F_q^{(NLO)}(R, L) = -\frac{\pi^2 L}{1152\ \sigma R^3}\left[2E_4(\tau) - E_2^2(\tau)\right]$$

$$+O\left(1/(\sigma L R)^2\right)$$

where $E_2$ and $E_4$ are the second and fourth order Eisenstein functions.

$$
\begin{aligned}
E_2(\tau) &= 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n && (1)\\
E_4(\tau) &= 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n && (2)\\
q &\equiv e^{2\pi i\tau} \quad , && (3)
\end{aligned}
$$

where $\sigma(n)$ and $\sigma_3(n)$ are, respectively, the sum of all divisors of $n$ (including 1 and $n$), and the sum of their cubes.

# Comparison with MC simulations

We studied the 3d gauge Ising model at

$\beta = 0.75180$ which corresponds to $T_c = 8$

For this value of $\beta$ the string tension is known with very high precision $\sigma = 0.018943(32)$. We performed simulations at $T = \frac{T_c}{3}, \frac{T_c}{2}, \frac{2T_c}{3} \frac{4T_c}{5}$ corresponding to $L = 24, 16, 12, 10$ respectively. For a wide range of values of $R$ we studied the ratios

$$Q_q(R) \equiv \ln \frac{G(R)}{G(R+1)} - \sigma_0(\beta) L$$

which are chosen so as to cancel the classical part of the free energy.

$$Q_q(R) = F_q(R+1, L) - F_q(R, L)$$

# Results

- As $T$ decreases the data move toward the free bosonic string prediction.

- However for $T \geq T_c/3$ the data disagree within the errors with the simple "Lüscher" term (i.e. the result obtained neglecting string self-interactions). The disagreement increases as $T$ increases.

- For all values of $T$ MC data agree rather well with the Nambu-Goto string correction truncated at the first perturbative order.

- The agreement becomes worse as $R$ decreases and the flux tube thickness $R = R_c = 8$ is approached.

- We tested also other values of $\beta$, finding the same results. The data show a very good scaling behaviour

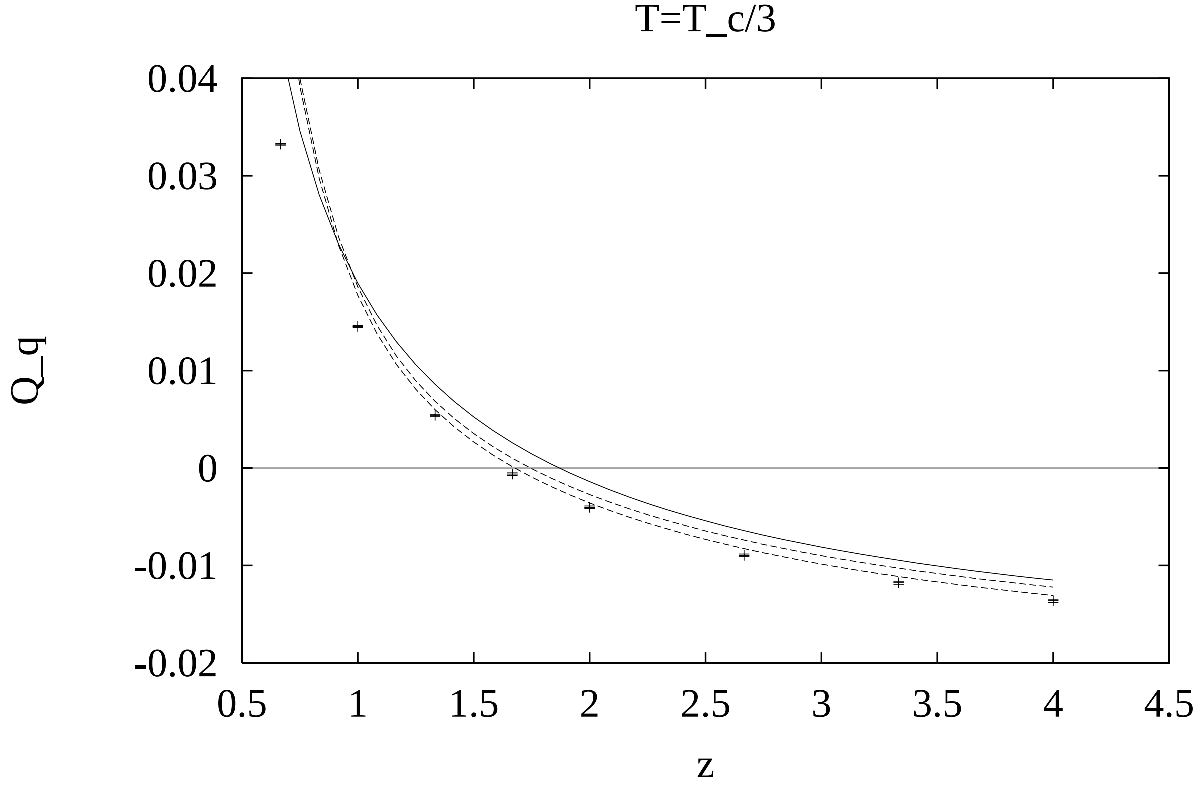



Figure 3: $Q_q$ for $N_t = 24$ (i.e. $T = T_c/3$) at $\beta = 0.75180$. The continuous line corresponds to the free bosonic string prediction, while the two dashed lines correspond to the first Nambu-Goto corrections. The difference between the two dashed lines keeps into account the uncertainty in the estimate of $\sigma$

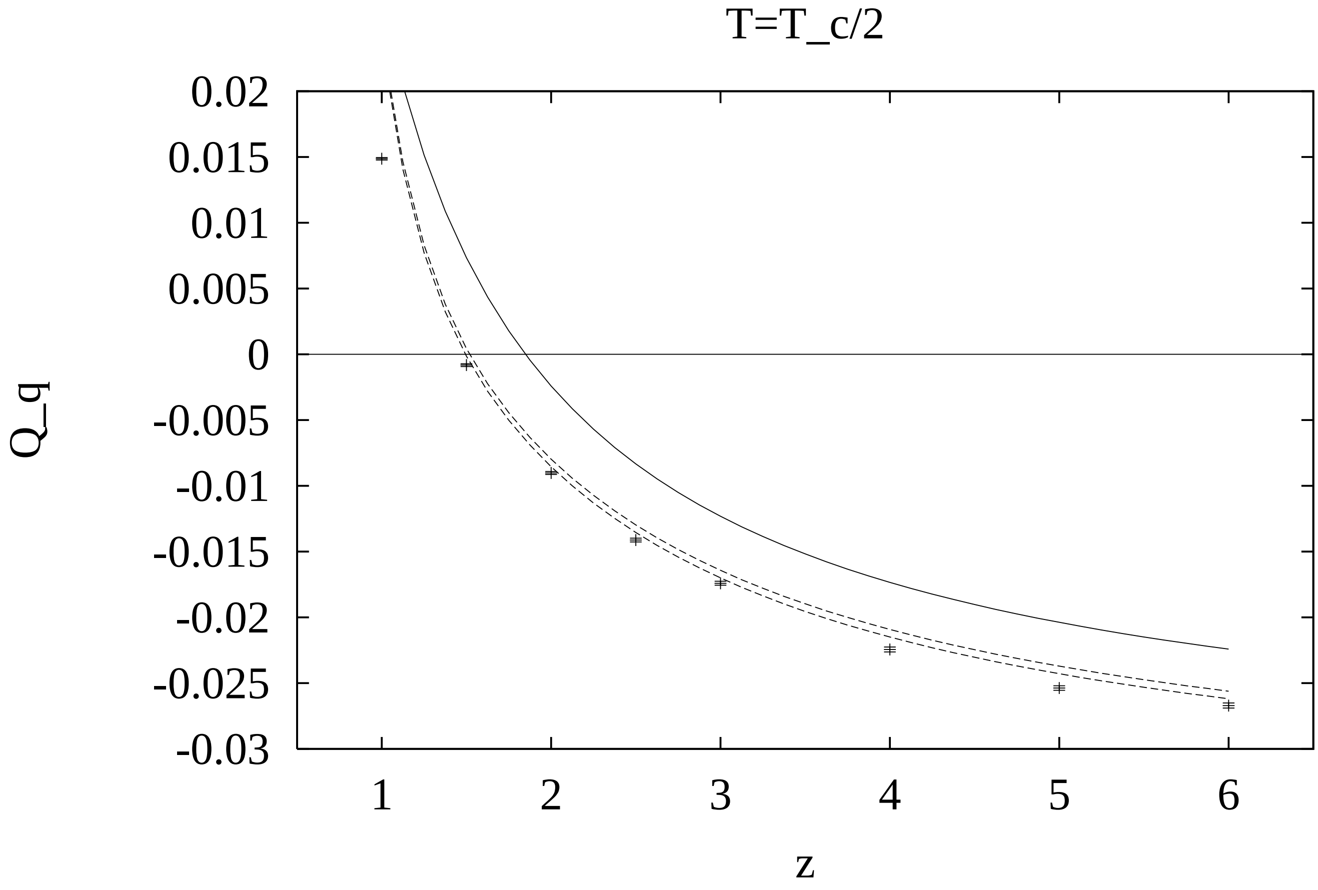



Figure 4: Same as above, but for $N_t = 16$ (i.e. $T = T_c/2$)

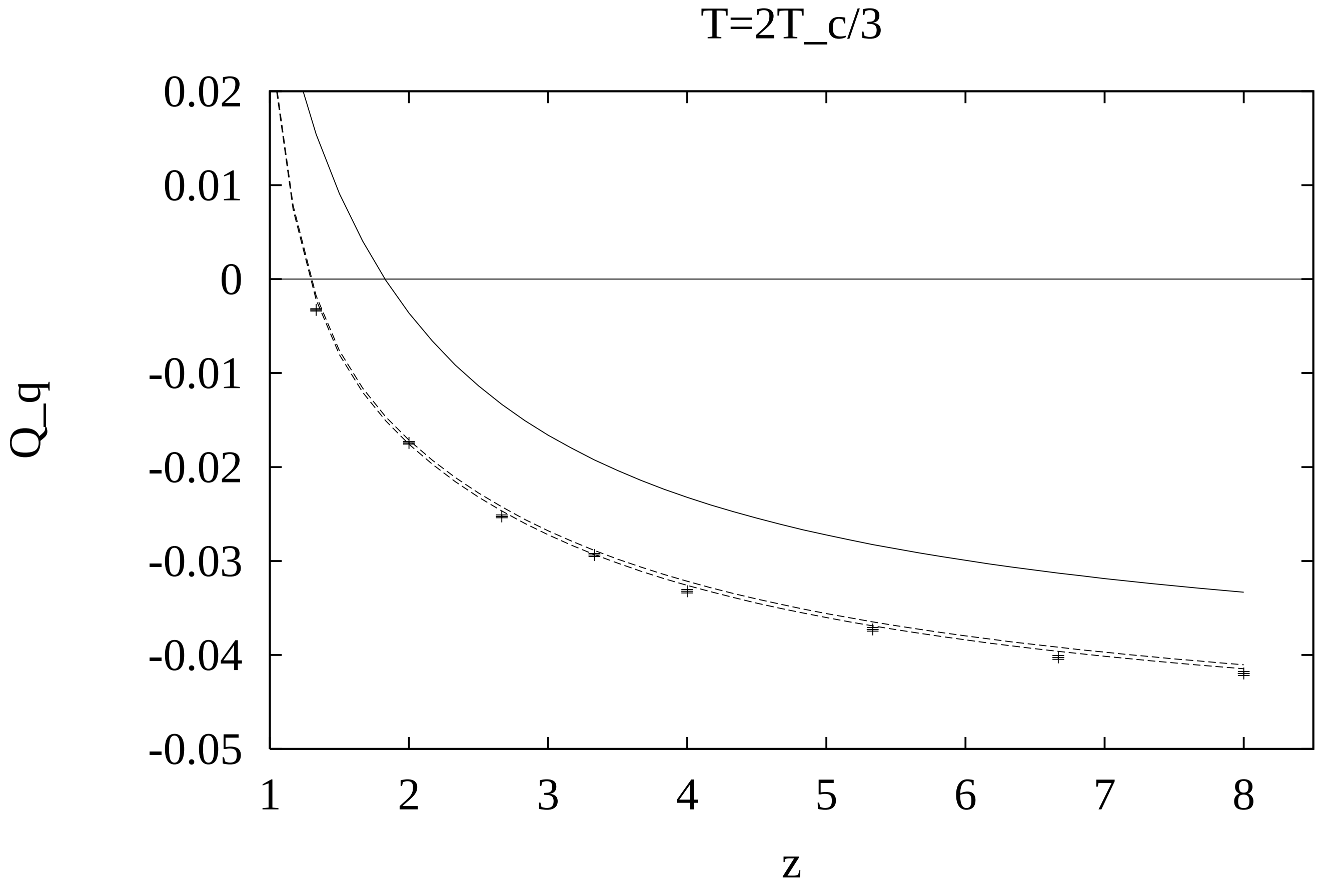



Figure 5: Same as above, but for $N_t = 12$ (i.e. $T = 2T_c/3$)

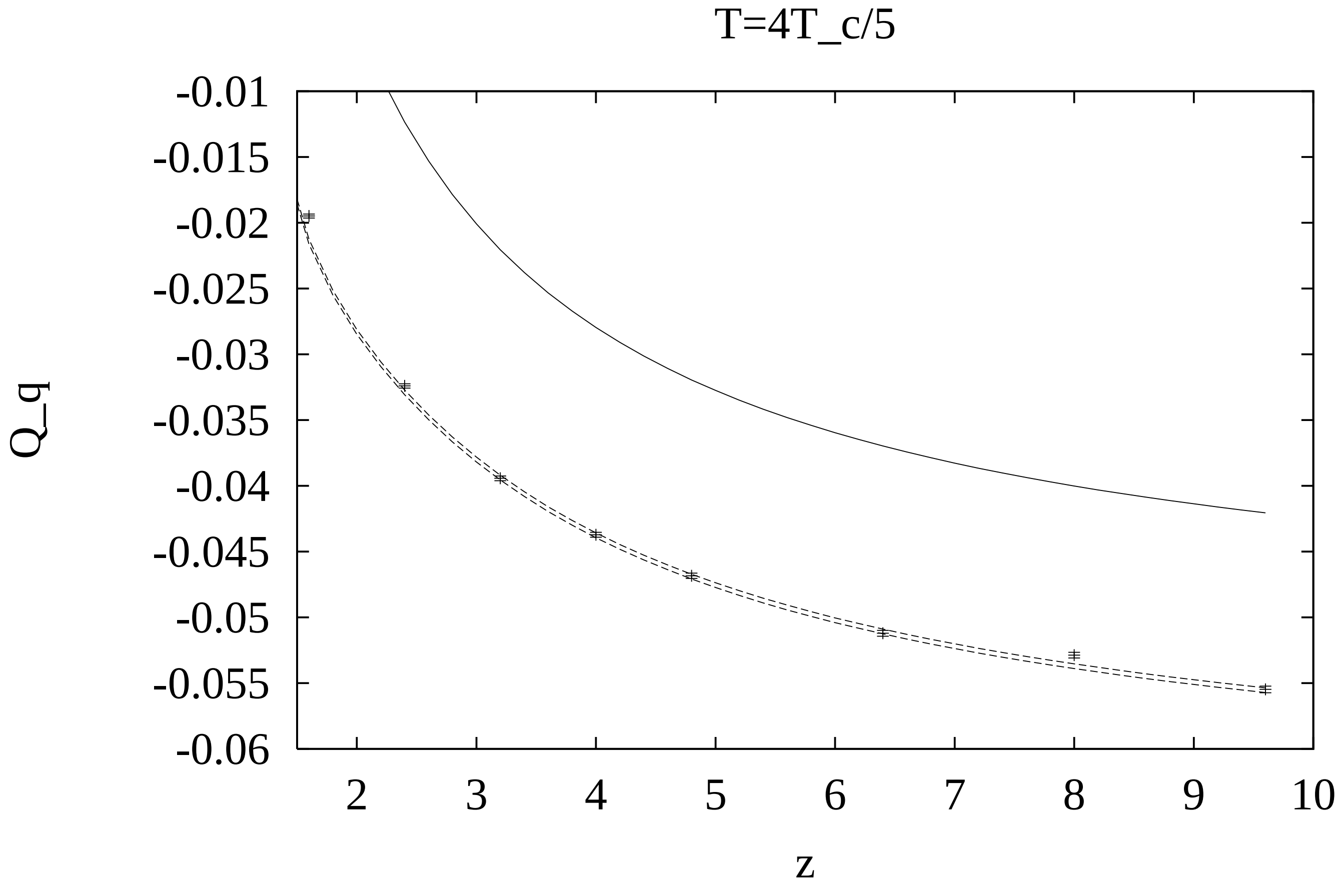



Figure 6: Same as above, but for $N_t = 10$ (i.e. $T = 4T_c/5$)

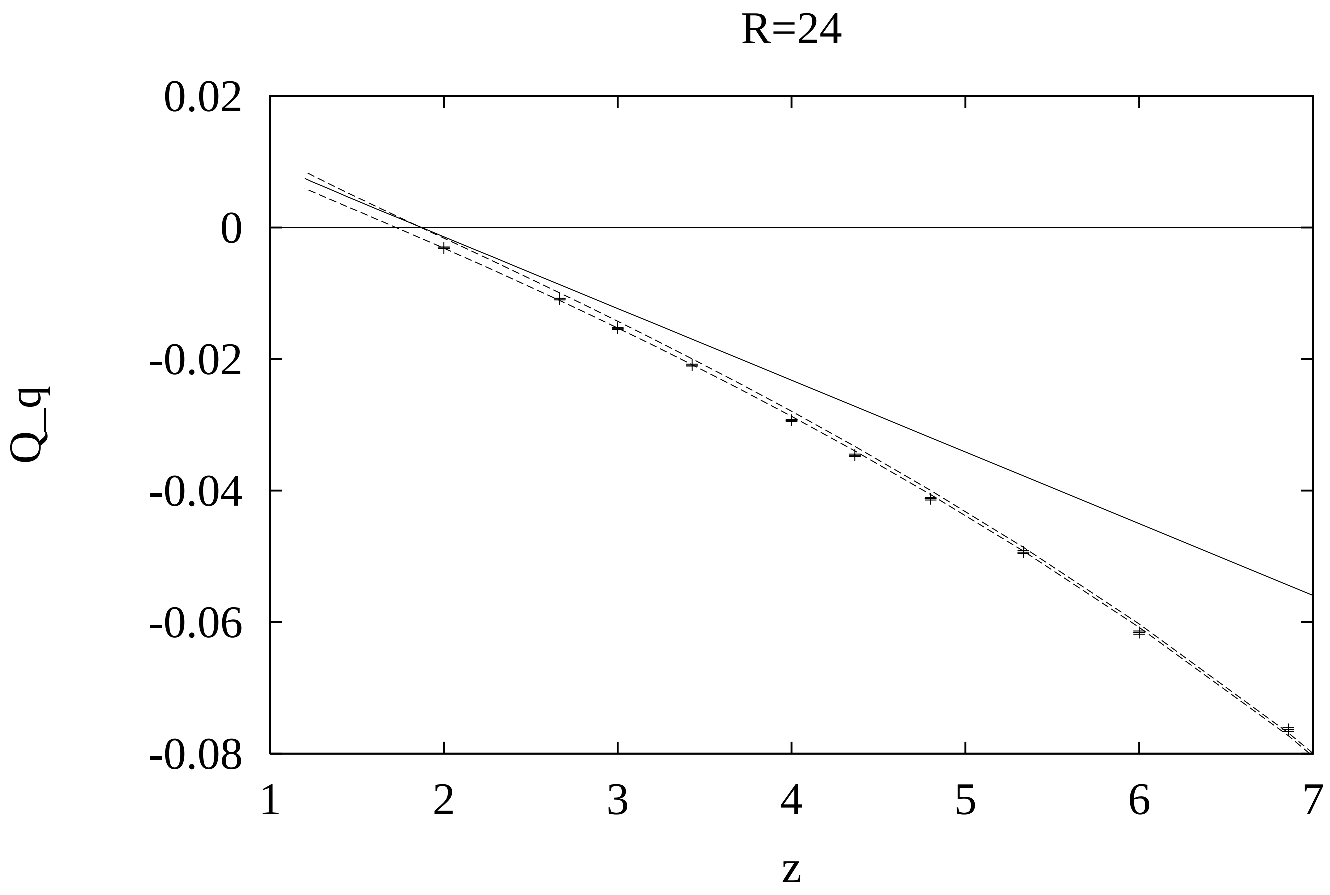



Figure 7: Same as above, but for a fixed $R = 24$

# Comparison with $SU(3)$ LGT

The data reported in

*M. Lüscher and P. Weisz,*
*"Quark confinement and the bosonic string"*
*JHEP 0207 (2002) 049.*

for SU(3) in d=3 and d=4 perfectly agree with free bosonic string prediction in $d = 3$ and show a small correction in $d = 4$. Notice however that the data are extracted in a different regime $(T \sim T/10)$

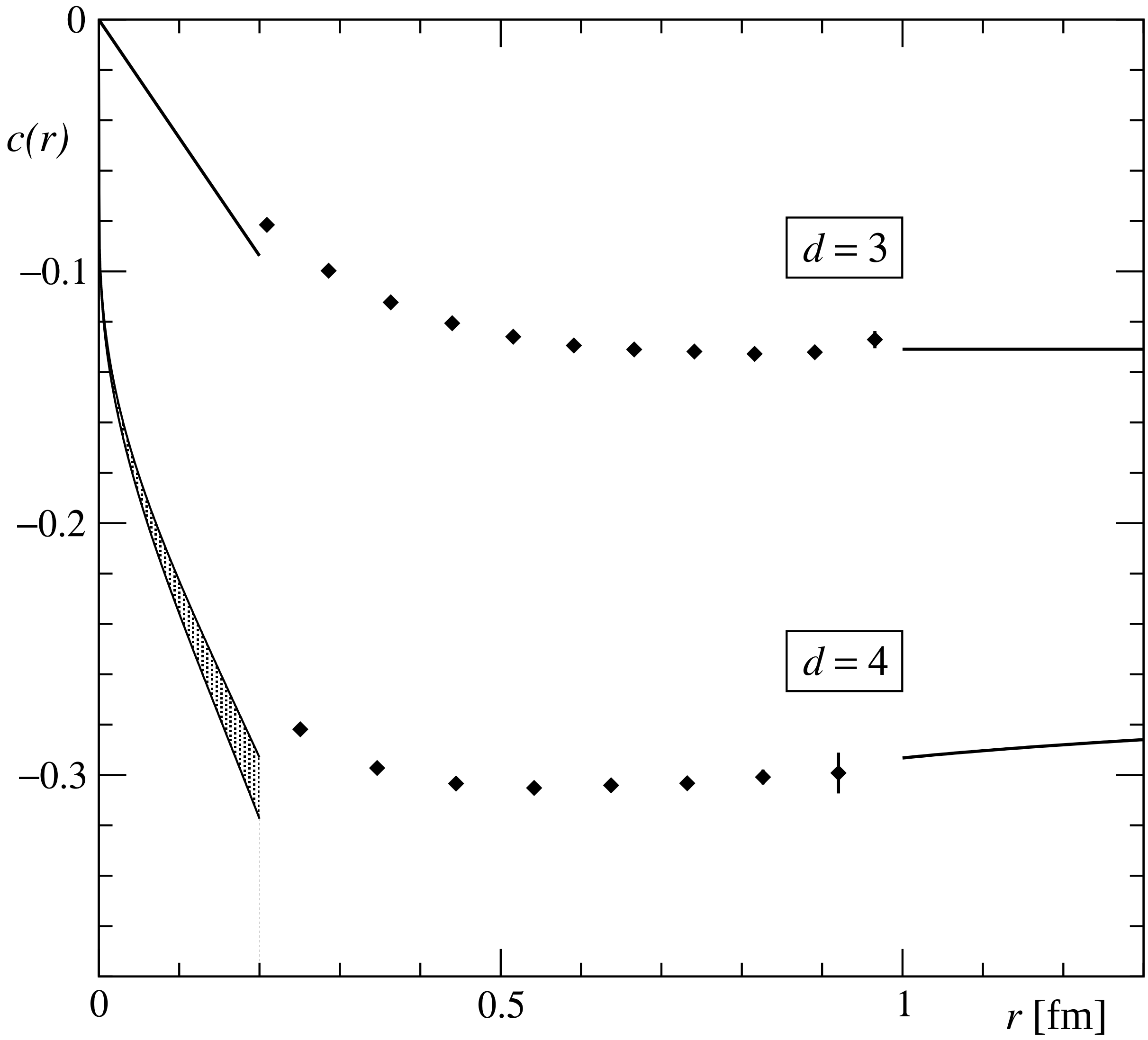



Figure 8: Slope $c(r)$ in three and four dimensions at $\beta = 20.0$ and $\beta = 6.0$ respectively. The curves on the left are obtained from perturbation theory, while those on the right derive from the string theory formula (3.9) with $b = 0$ and $b = 0.04$ fm.

# Conclusions

- MC simulations strongly support the conjecture that the effective string theory which describes confinement in the infrared regime of various LGT's (3d gauge Ising model, SU(2), SU(3) in d=3 and d=4) at large enough distances and low enough temperatures is a simple free bosonic string theory.

- At smaller distances and/or higher temperatures the effective string picture still holds, but corrections, presumably due to the self-interaction terms in the string action, appear.

- The peculiar geometry of the Polyakov loop correlators implies that they are perfect tools to explore this regime. In particular at intermediate temperatures (say, $T \geq T_c/3$) higher order effects (which encode the self-interaction of the bosonic fields) become important and cannot be neglected.

- This fact can be used to better understand the nature of the underlying effective string. In principle there could be various different classes of effective string theories. All of them with the same large distance limit (the free bosonic string) but different actions

- As a first step in this direction we notice that MC simulations for the 3d gauge Ising model, are in very good agreement with a "Nambu-Goto" form of these self-interaction terms, truncated at the first perturbative order.

- Recent results by Lüscher and Weisz for SU(3) in d=3 and 4 show a different behaviour suggesting that these two theories could belong to two other classes of effective string models

# From the Nambu-Goto action to the free bosonic string.

The Nambu string action is given by the area of the world–sheet:

$$S = \sigma \int_0^T d\tau \int_0^R d\varsigma \sqrt{g} \quad ,$$

where $g$ is the determinant of the two–dimensional metric induced on the world–sheet by the embedding in $R^d$:

$$\begin{aligned} g = \det(g_{\alpha\beta}) \quad &= \quad \det \; \partial_\alpha X^\mu \partial_\beta X^\mu \quad . \\ & \qquad (\alpha, \beta = \tau, \varsigma, \;\; \mu = 1, \ldots, d) \end{aligned}$$

and $\sigma$ is the string tension.

The reparametrization and Weyl invariances of the action require a gauge choice for quantization.

We choose the "physical gauge"

$$\begin{aligned} X^1 &= \tau \\ X^2 &= \varsigma \end{aligned}$$

so that $g$ is expressed as a function of the transverse degrees of freedom only:

$$\begin{aligned} g &= 1 + \partial_\tau X^i \partial_\tau X^i + \partial_\varsigma X^i \partial_\varsigma X^i \\ &\quad + \partial_\tau X^i \partial_\tau X^i \partial_\varsigma X^j \partial_\varsigma X^j - (\partial_\tau X^i \partial_\varsigma X^i)^2 \\ &\quad (i = 3, \ldots, d) \ . \end{aligned}$$

The fields $X^i(\tau, \varsigma)$ satisfy Dirichlet boundary conditions on $M$:

$$X^i(0, \varsigma) = X^i(T, \varsigma) = X^i(\tau, 0) = X^i(\tau, R) = 0 \quad .$$

Due to the Weyl anomaly this gauge choice can be performed at the quantum level only in the critical dimension $d = 26$. However, the effect of the anomaly is known to disappear at large distances, which is the region we are interested in.

Expanding the square root in the action we obtain, discarding terms of order $X^4$ and higher

$$\begin{aligned} S &= \sigma R T + \frac{\sigma}{2} \int d^2\xi X^i (-\partial^2) X^i \\ \partial^2 &= \partial_\tau^2 + \partial_\varsigma^2 \ . \end{aligned}$$

# The partition function of the free bosonic string.

The partition function for the free bosonic action is

$$Z_q(R,T) \propto \left[\det(-\partial^2)\right]^{-\frac{d-2}{2}} \ .$$

The determinant must be evaluated with Dirichlet boundary conditions.

The spectrum of $-\partial^2$ with Dirichlet boundary conditions is given by the eigenvalues

$$\lambda_{mn} = \pi^2 \left(\frac{m^2}{T^2} + \frac{n^2}{R^2}\right)$$

corresponding to the normalized eigenfunctions

$$\psi_{mn}(\xi) = \frac{2}{\sqrt{RT}} \sin\frac{m\pi\tau}{T} \sin\frac{n\pi\varsigma}{R} \ .$$

The determinant can be regularized with the $\zeta$-function technique: defining

$$\zeta_{-\partial^2}(s) \equiv \sum_{mn=1}^{\infty} \lambda_{mn}^{-s}$$

the regularized determinant is defined through the analytic continuation of $\zeta'_{-\partial^2}(s)$ to $s = 0$:

$$\det(-\partial^2) = \exp\left[-\zeta'_{-\partial^2}(0)\right] \quad .$$

The series can be transformed, using the Poisson summation formula which states that:

$$\sum_{n=-\infty}^{\infty} f(nr) = \frac{1}{r} \sum_{m=-\infty}^{\infty} \tilde{f}\left(\frac{m}{r}\right)$$

where the Fourier transform $\tilde{f}$ is defined as

$$\tilde{f}(y) = \int_{-\infty}^{\infty} dx \; e^{-2\pi i x y} f(x)$$

we obtain:

$$\zeta_{-\partial^2}(s) = -\frac{1}{2}\left(\frac{R^2}{\pi^2}\right)^s \zeta_R(2s) +$$

$$\frac{\sqrt{\pi} Im\tau \Gamma(s-1/2)}{2\Gamma(s)}\left(\frac{R^2}{\pi^2}\right)^s \zeta_R(2s-1) +$$

$$\frac{2\sqrt{\pi}}{\Gamma(s)}\left(\frac{T^2}{\pi^2}\right)^s \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\left(\frac{\pi p}{n Im\tau}\right)^{s-1/2} K_{s-1/2}(2\pi p n Im\tau)$$

where $\tau = iT/R$, $\zeta_R(s)$ is the Riemann $\zeta$ function and $K_\nu(x)$ is a modified Bessel function. The derivative $\zeta'_{-\partial^2}(s)$ can be analytically continued to $s = 0$ where it is given by

$$\zeta'_{-\partial^2}(0) = \log(\sqrt{2}R) - \frac{i\pi\tau}{12} - \sum_{n=1}^{\infty}\log(1-q^n)$$

where we have defined

$$q \equiv e^{2\pi i\tau} \ .$$

Introducing the Dedekind $\eta$-function

$$\eta(\tau) = q^{1/24}\Pi_{n=1}^{\infty}(1-q^n)$$

we obtain finally

$$\det(-\partial^2) = \exp[-\zeta'_{-\partial^2}(0)] = \frac{\eta(\tau)}{\sqrt{2R}}$$

and

$$Z_q(R,T) \propto \left[\frac{\eta(\tau)}{\sqrt{R}}\right]^{-\frac{d-2}{2}} .$$