Matrix Models and QCDwith Chemical Potential

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coworker:

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Plan:

1. **Motivation**
2. **The concept of Matrix Models (MM)**
3. **Chiral symmetry breaking (χSB) & Dirac spectrum**
4. **MM = ε-regime of chiral Perturbation Theory ($\varepsilon\chi$PT)**
5. **Comparison to Lattice results**
6. **Conclusions**

1. Motivation:

- **QCD** under extreme conditions = finite Temperature & density
  fascinating theoretically + experimentally

- **sign problem:**
  $\not{D} = \gamma_\nu^E(\partial_\nu + iA_\nu) \; = \; -\not{D}^\dagger$ for $\mu = 0$
  $\not{D} + \gamma_0^E \mu$ **finite $\mu \neq 0$ breaks anti-hermiticity**
  $\Rightarrow$ $\det(\not{D} + \gamma_0 \mu)$ has **complex phase**, no importance sampling!
  but $\mathcal{Z}_{\mathbf{QCD}}$ and other observables remain real

- **quenched QCD+μ vs. unquenched:**
  in quenched approx. **chiral symetry unbroken** $\sim m_\pi$
  $\longrightarrow$ understand role of dynamical quarks, χPT μ–independent ?

- **SU(2) & adjoint**: a way out
  – $\det(\not{D} + \gamma_0 \mu)$ remains real $\longrightarrow$ Lattice simulations

- **Effective Field theory & Matrix Model:**

– based on **global symmetries** & **exactly solvable**

∃ **2 types of MM:** (not mutually exclusive)

(a) **phenomenological** à la Landau-Ginsburg)

b) **"exact" limit of QCD** $= \varepsilon\chi$**PT**

What can we learn from MM?

- which theory: **QCD**, $SU(2)$, adj.rep. $\Rightarrow$ a) **phase structure**

- input SχSB $\langle\bar{q}q\rangle \neq 0$

$\Rightarrow$ b) extract **low energy constants** $\langle\bar{q}q\rangle$, f_π ($\mu \neq 0!$) on Lattice:

from $\not{D}$ eigenvalue spectrum ≈ 0 dependence on

- **chemical potential** μ
- **quark masses** m_{quark}
- **number of flavours** N_f
- **topology** ν

- **Localise sign problem**

- **Limitations:**

  N_f flavour dependence too weak for

  – $\beta(g)$ change sign,

  – N_f dependence of order of phase transition

2. The Matrix Model Approach

• **idea:** replace theory Hamiltonian, $\not{D}$, ... by matrix with independent entries (Gauss) of **same global symmetry**: time-reversal + spin (Wigner-Dyson)

• anti-hermitian $\not{D} \longrightarrow \begin{pmatrix} 0 & \Phi \\ -\Phi^\dagger & 0 \end{pmatrix}$, Φ random matrix $\in \mathbb{R} \,/\, \mathbb{C} \,/\, \mathbb{H}$

for **SU(2)/ QCD / adjoint**

and $\langle \cdots \rangle_{GAUGE} \longrightarrow \langle \cdots \rangle_{MM}$ [Shuryak, Verbaarschot '93]

[Verbaarschot '94]

+ **fixed topology:** $\Phi = \boxed{}$

• **chemical potential**

$\not{D} + \gamma_0 \mu_B$
non-Hermitian

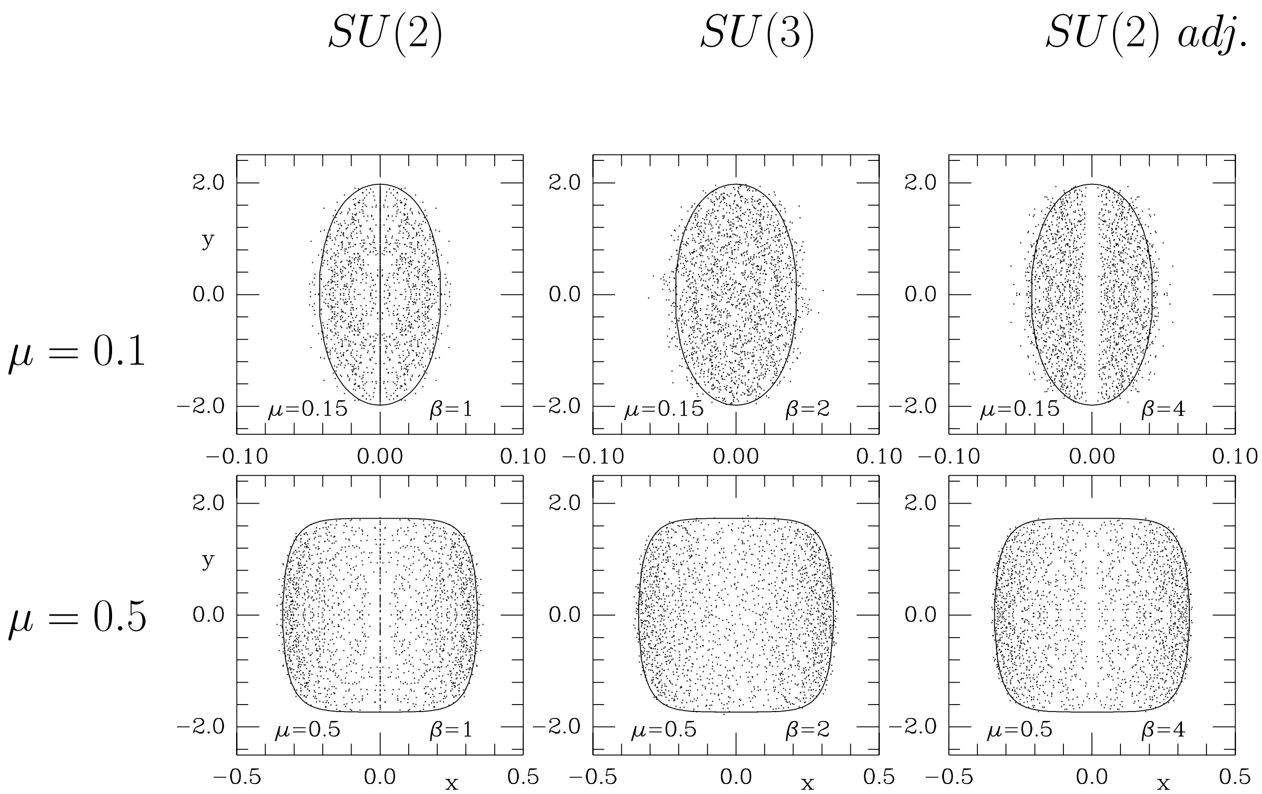
[Stephanov '96; Halasz, Osborn Verbaarschot '97]

Anti-Unitary symmetries = 3-fold way

$*\ \{D\!\!\!/, \gamma_5\} = 0 \ \Rightarrow$ **block structure** + [Verbaarschot '94]:

• $SU(2)$ fund.	• $SU(N_c \geq 3)$ fund.	• $SU(N_C)$ adjoint
$D\!\!\!/ = \begin{pmatrix} 0 & \Phi \\ -\Phi^T & 0 \end{pmatrix}$	$D\!\!\!/ = \begin{pmatrix} 0 & \Phi \\ -\Phi^\dagger & 0 \end{pmatrix}$	$D\!\!\!/ = \begin{pmatrix} 0 & \Phi \\ -\Phi^\dagger & 0 \end{pmatrix}$
ϕ-elements $\in \mathbb{R}$	$\in \mathbb{C}$	$\in$ real $\mathbb{H}$
from $[C\sigma_2 K, D\!\!\!/] = 0$		**from** $[CK, D\!\!\!/] = 0$
$SU(N_C)$ adjoint staggered	also staggered	$SU(2)$ fund. staggered

Scatterplots of complex MM eigenvalues



[M.A. Halasz, J.C. Osborn, J.J.M. Verbaarschot Phys.Rev. D56 (97) 7059]

3. Relation $\not{D}$ spectrum $\leftrightarrow$ χSB $(\mu = 0)$

$*\ \ \mu = 0$: Euclidean + finite V

$$\not{D} q_k = i\lambda_k q_k \qquad \lambda_k \in \mathrm{R} \ \text{ for } \mu = 0$$

$$\rho(\lambda) \equiv \langle \, \Sigma'_k \, \delta(\lambda - \lambda_k) \, \rangle$$ **spectral density**

chiral symmetry

$$\{\not{D}, \gamma_5\} = 0 \quad \Rightarrow \rho(\lambda) = \rho(-\lambda)$$

$$\nu = |n_L - n_R|, \ \ \lambda_k = 0 \leftrightarrow \text{topology}$$

Banks-Casher

$$\boxed{\rho(\mathbf{0}) = V|\langle\bar{q}q\rangle|/\pi}$$

from **Resolvent:**

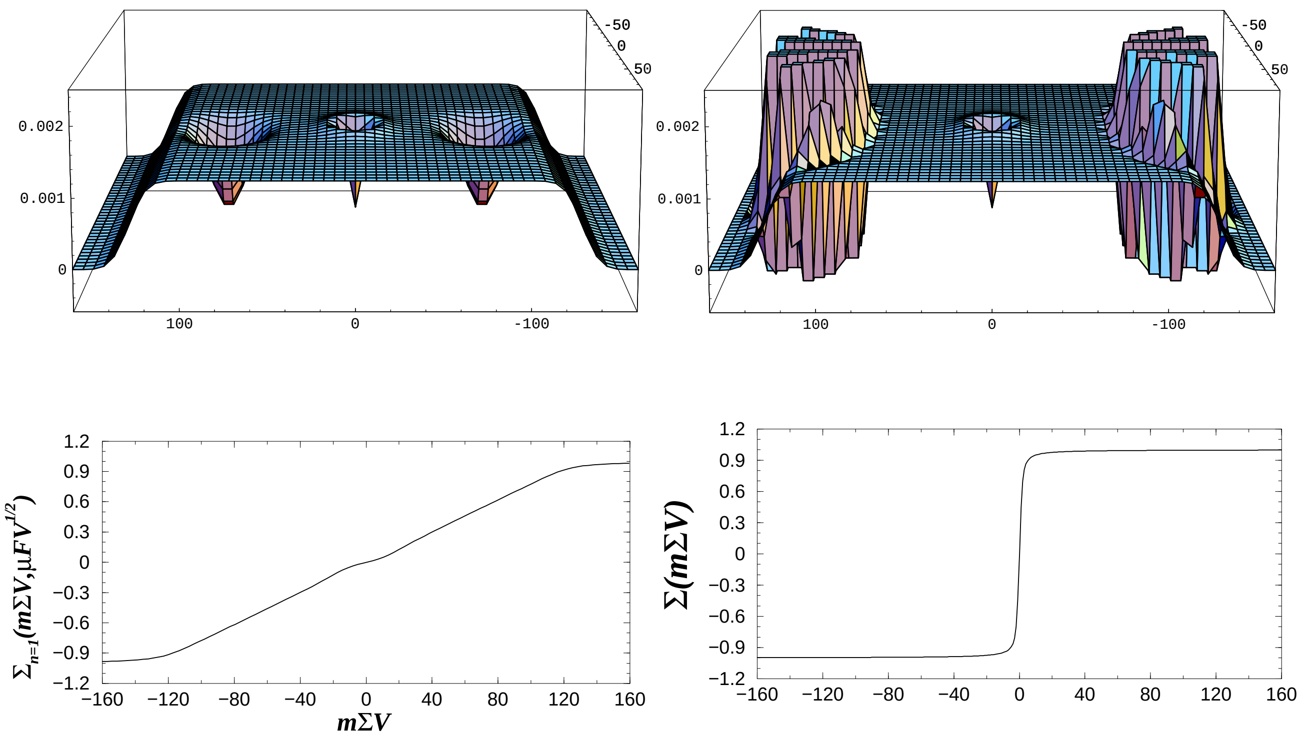
$$\Sigma(m) \equiv -\frac{1}{V}\,\partial_m \log \mathcal{Z}_{\mathbf{QCD}} = -\frac{1}{V} \int d\lambda \rho(\lambda) \frac{2m}{(\lambda^2 + m^2)}$$

invert for $\rho(\lambda) = \frac{1}{2\pi}[\Sigma(i\lambda + \varepsilon) - \Sigma(i\lambda - \varepsilon)]$ (if m-independent)

- χ**SB:** $\exists$ jump at $\lambda = 0$

$*\ \ \mu \neq 0$: $\int d^2z$, No $\delta^{(2)}(z)$ when $\lim m \to 0$! **still jump?**

χSB at $\mu_B \neq 0$:



• **phase quenched** $N_f = 2$ vs. **unquenched** $N_f = 2$:
complex spectral density $\rho(z)$ (upper) vs. num. resolvent $\Sigma(m)$ (lower):
$\exists$ **oscillatory region** $\Rightarrow$ χ**SB**

[J.C. Osborn, K. Splittorff, J.J.M. Verbaarschot Phys.Rev.Lett. 94 (05) 202001; hep-lat/0510118]

4. From QCD to χPT to MM

$\mathcal{Z}_{\mathbf{QCD}}$	$\longrightarrow$	$\mathcal{Z}_{\chi PT}$	$\longrightarrow$	$\mathcal{Z}_{\varepsilon}$
q,A	$\langle\bar{q}q\rangle \neq 0$	pions $\pi(x)$	$\lvert V\rvert < \infty$	$SU(N_f)$ integral
		"Goldstone"	0-mode	$\Leftrightarrow \mathcal{Z}_{MM}$

box $V = L^4$ $\boxed{1/\Lambda \ll L \ll 1/m_\pi}$

1) **low momenta** $1/L \ll \Lambda$ non-Goldstone scale

2) **Compton wavelength** $1/m_\pi \gg L$ box size

[Gasser, Leutwyler 87; Leutwyler, Smilga 92]

Solution of MM: σ-model

- $\mathcal{Z}_{MM} = \int d\phi_{N\times N} \prod_f^{N_f} \det[\not{D} + \gamma_0\mu_f + m_f] \exp[-N\langle\bar{q}q\rangle^2 \mathrm{Tr}\, \phi\phi^\dagger]$

– $\det[\ldots] = \int d\psi d\overline{\psi} \exp[\overline{\psi} \ldots \psi] \quad \longrightarrow \quad$ do Gaussian $\int d\phi$

– **Hubbard-Stratonovich**: extra $\int dQ\ N_f \times N_f\ \rightarrow$ do $\int d\psi d\overline{\psi}$

$$\Rightarrow\ \mathcal{Z}_{MM} \sim \int dQ_{N_f\times N_f} \det[Q^\dagger]^\nu \det\left[Q^\dagger Q - \mu^2 Q^\dagger \Sigma_3 Q^{\dagger\,-1}\Sigma_3\right]^{\mathbf{N}}$$

$$\times \mathrm{e}^{-N\langle\bar{q}q\rangle^2 \mathrm{Tr}\left(Q^\dagger Q\ -\ M(Q+Q^\dagger)\right)}$$

$\mu_f = \pm\mu$ and **Saddle Point** $\mathbf{N} \rightarrow \infty$:

- **MM** $= \varepsilon\chi$PT group integral

advantage of MM: all density correlations easy

from **orthogonal polynomials** $\in\ \mathbb{C}$ [G.A. et al., Osborn 04]
& Replicas [Splittorff, Verbaarschot 03+04]

Chiral Perturbation Theory & the ε-regime $(\mu = 0)$

$$\mathcal{Z}_{\chi PT} \equiv \int_{SU(N_f)} [dU(x)] \, \exp[-\int \mathrm{Tr}\mathcal{L}_{eff}(U, \partial U)]$$

$$\mathcal{L}_{eff} = \frac{f_\pi^2}{4} \partial U(x) \partial U(x)^\dagger - \frac{1}{2}\langle\bar{q}q\rangle (M_f \mathrm{e}^{i\frac{\Theta}{N_f}} U(x) + (M_f \mathrm{e}^{i\frac{\Theta}{N_f}} U(x))^\dagger) + ..$$

parametrise $U \in SU(N_f)$ for **QCD**, else $U \Sigma U^T$ Σ metric

- **ε-regime** = const. U_0 dominate $U = U_0 \, \mathrm{e}^{\xi(x)}$ for $m_\pi \sim \frac{1}{L^2} \ll \frac{1}{L}$

$$\mathcal{Z}_\varepsilon \equiv \int_{U(N_f)} dU_0 \det[U_0]^\nu \, \exp[-\frac{1}{2}\langle\bar{q}q\rangle V \mathrm{Tr}\left(M_f(U_0 + U_0^\dagger)\right)] \times \text{free fields}$$

after fixing ν topology: invert $\mathcal{Z}_{\mathbf{QCD}} = \sum_{\nu=-\infty}^{+\infty} \exp[i\Theta\nu] \, \mathcal{Z}_\nu(M)$

- **analytic solution** $\sim$ det [Bessel-$I(m_f V \langle\bar{q}q\rangle)$'s] $\Leftrightarrow$ **MM** (**QCD**)

[. . .]

$\varepsilon\chi$PT at $\mu \neq 0$:

- **add** $\mu^2 V \ll 1/L^2$ for U_0 dominates

$$\mathcal{Z}_\varepsilon(\mu) \;=\; \int dU_0 \det[U_0]^\nu \; \mathrm{e}^{-\frac{1}{2}\mu^2 f_\pi^2 V \mathrm{Tr}(U_0\, B\, U_0^\dagger B) \;+\; \frac{1}{2} V \langle \bar{q} q\rangle \mathrm{Tr}(M_f (U_0 + U_0^\dagger))}$$

- **scaling**: $\mu^2 f_\pi^2 V$ and $V M_f \langle \bar{q} q \rangle$
- μ-independent for Baryon charge $B = 1\!\!|$ (quarks only)
- equivalent to complex MM (**QCD**): [Splittorff,Verbaarschot 03, G.A., Fyodorov, Vernizzi 04]

- **free MM parameter** $= f_\pi$
- **Replica** generating functional: **conjugate quarks** $B \neq 1\!\!|$

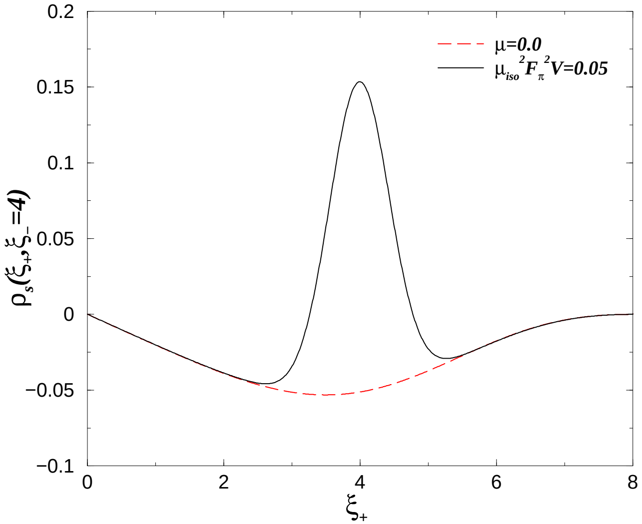
Partial quenching: imaginary μ

- $\varepsilon\chi$PT: **same** group integral with $\mu^2 \;\rightarrow\; -\mu^2$

- 2MM: $\not{D}_J + i\mu_J$, $J = 1, 2$ **eigenvalues** $X \neq Y$ **different, real**

$$\rho(X;\ Y) = \langle \mathrm{Tr}\delta(\not{D}_1(\mu_1) - X)\mathrm{Tr}\delta(\not{D}_2(\mu_2) - Y)\rangle$$

NOT $\quad \delta(X - Y)$



f_π **from real correlation functions**, also $\mu_1 \neq 0 = \mu_2$

[Damgaard, Heller, Splittorff, Svetitsky, Toublan 05+06; G.A., Damgaard, Osborn, Splittorff 06]

5. Complex MM vs. Lattice data at $\mu \neq 0$

Examples for analytic predictions:

- variables $V\langle\bar{q}q\rangle z = \xi$ and $\mu^2 V f_\pi^2 = \hat{\mu}^2$

- $\rho(\xi) = \frac{1}{\hat{\mu}^2}|\xi|^2 K_\nu\left(\frac{|\xi|^2}{2\hat{\mu}^2}\right) \mathrm{e}^{+\frac{1}{4\hat{\mu}^2}(\xi^2+\xi^{*\,2})} \int_0^1 dt\, \mathrm{e}^{-2t\hat{\mu}^2} J_\nu(2\sqrt{t}\,\xi) J_\nu(2\sqrt{t}\,\xi^*)$

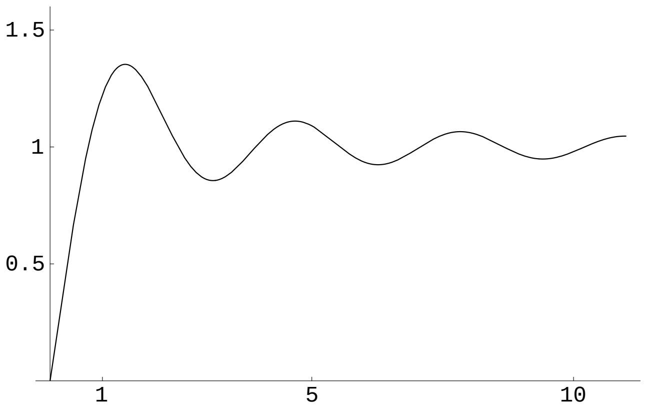
  quenched QCD [Splittorff, Verbaarschot 03]

- $\rho(\xi) = \frac{1}{32\hat{\mu}^4}(\xi^{*\,2} - \xi^2)\ |\xi|^2\ K_{2\nu}\left(\frac{|\xi|^2}{2\hat{\mu}^2}\right) \mathrm{e}^{+\frac{1}{4\hat{\mu}^2}(\xi^2+\xi^{*\,2})}$
  $\times \int_0^1 ds \int_0^1 \frac{dt}{\sqrt{t}}\, \mathrm{e}^{-2s(1+t)\hat{\mu}^2} \left(J_{2\nu}(2\sqrt{st}\ \xi) J_{2\nu}(2\sqrt{s}\ \xi^*)\ -\ (\xi \leftrightarrow \xi^*)\right)$

  quenched $SU(N_C)$ adj. / $SU(2)$ fund. staggered [G.A. 05]

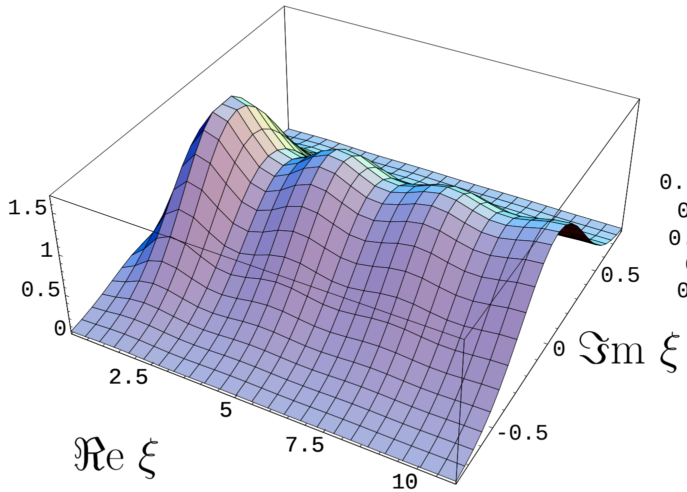
Real vs. complex QCD: $N_f = \nu = 0$

$\rho_{MM}(x)$

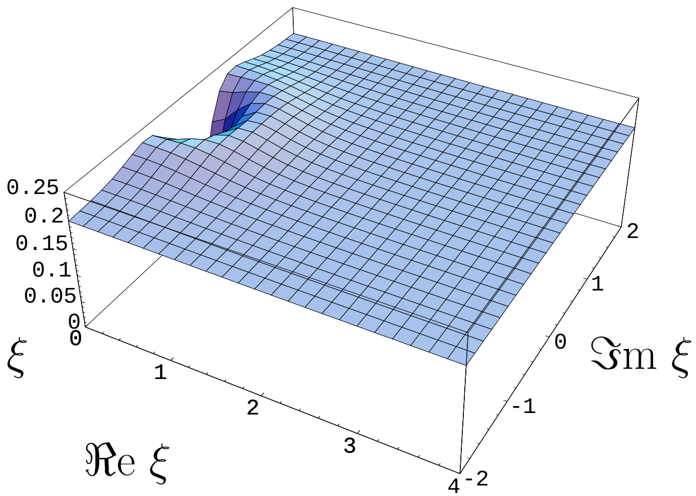


x

$\rho_{MM}(\xi = Nz)$

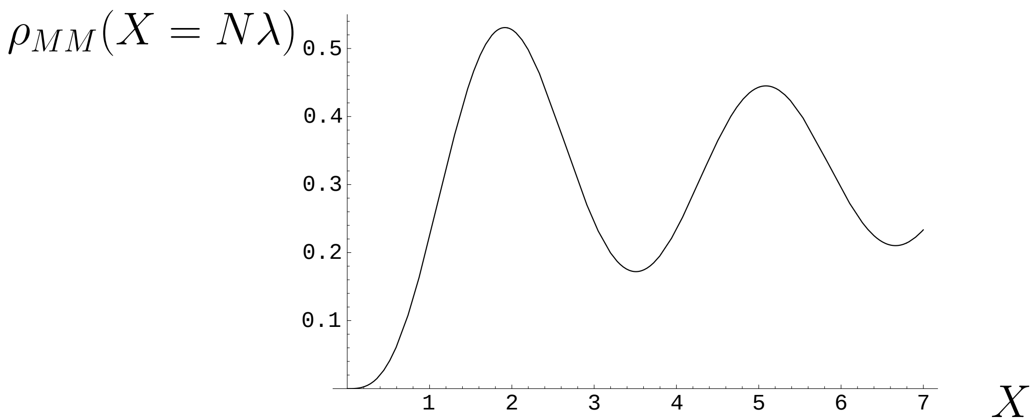


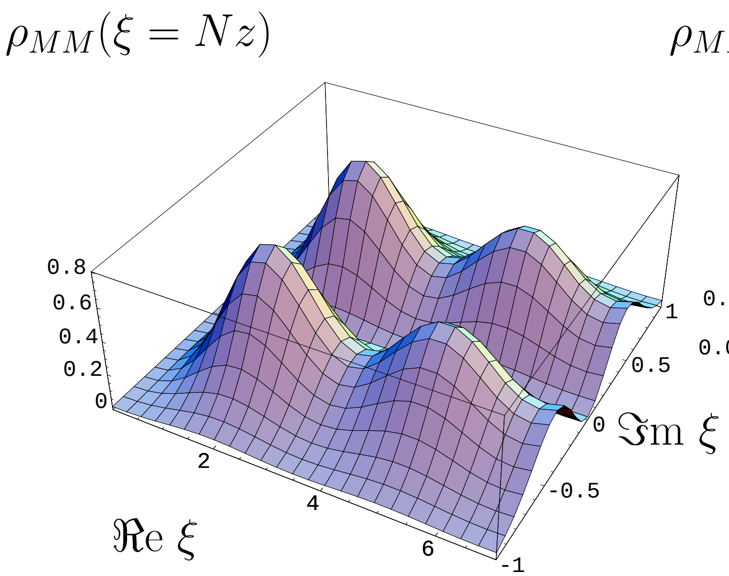
$\rho_{MM}(\xi = \sqrt{N}\, z)$

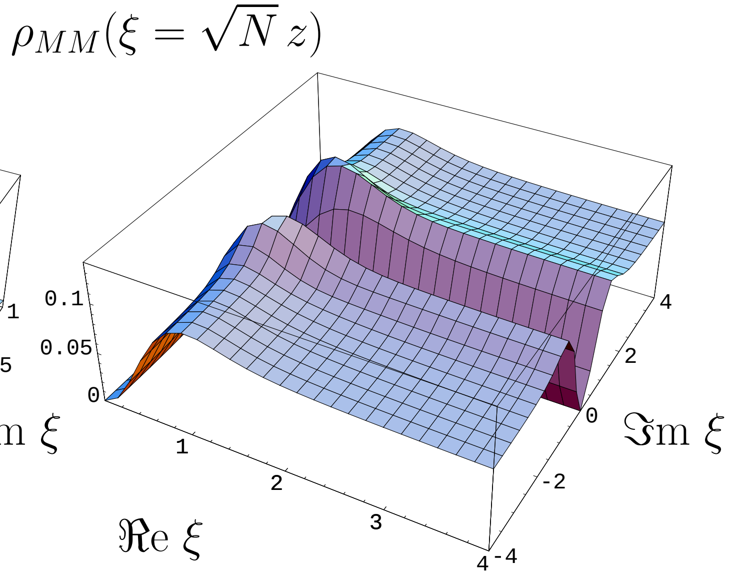


- **complex: small** $\mu^2 \ll 1$ (left) and $\mu \geq 1$ (right)

Real vs. complex $SU(N_C)$ adj. (=$SU(2)$ fund. stag.)

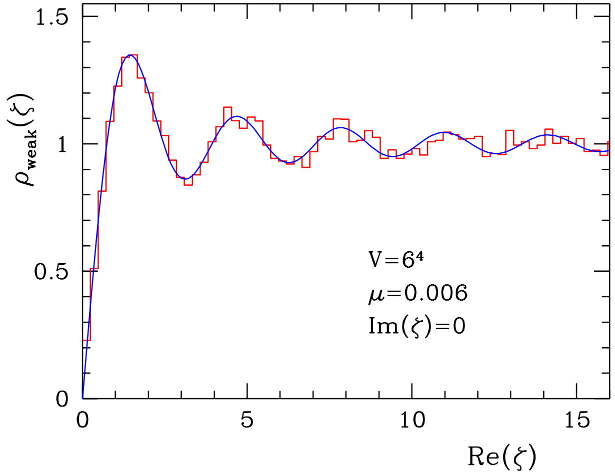


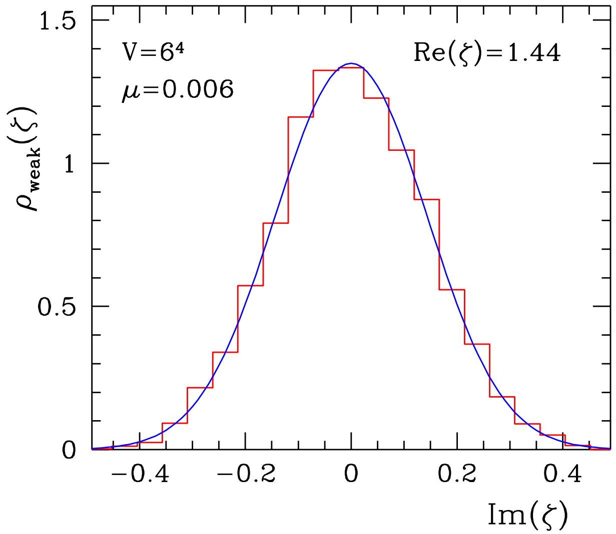




- **complex: small** $\mu^2 \ll 1$ (left) and $\mu \geq 1$ (right)

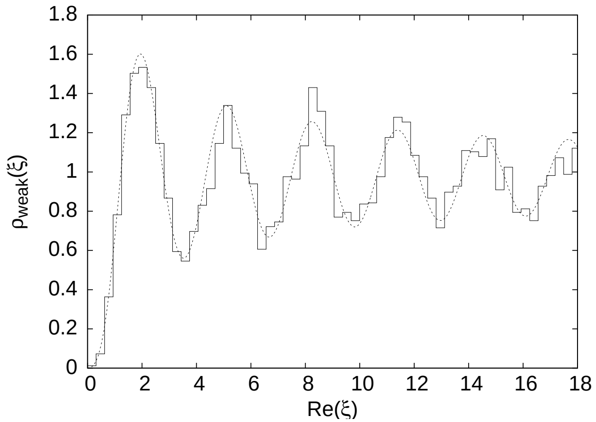
Comparison to quenched Lattice Data: small $\mu^2 V$

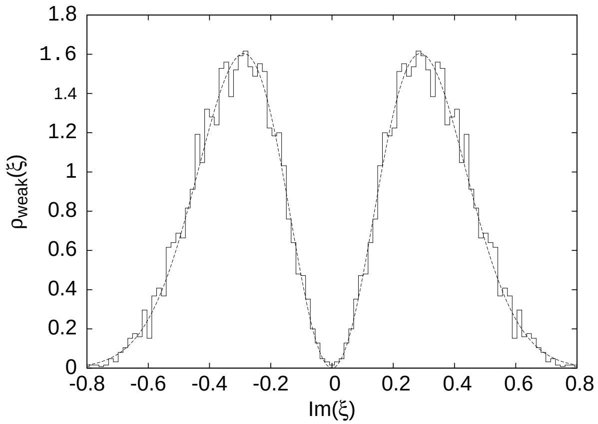




- **cuts** through the complex density for **QCD**:

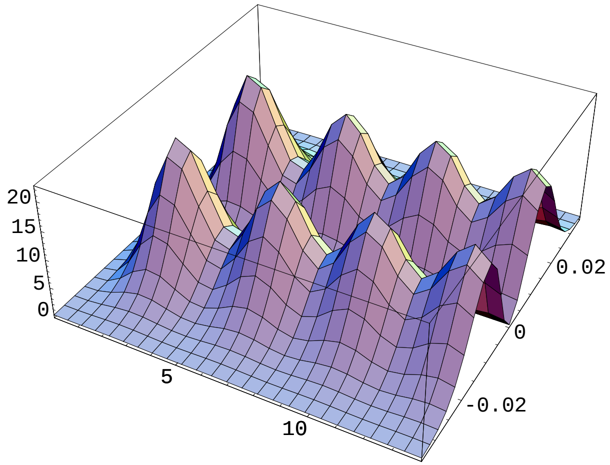
$\nu = 0$ staggered [G.A., T. Wettig 03]; $\nu \neq 0$ **overlapp** [Bloch, Wettig 06]

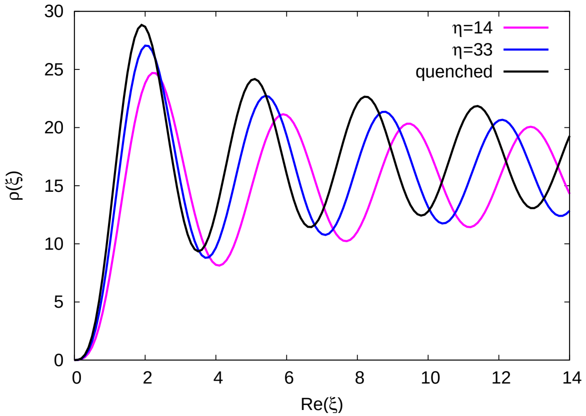




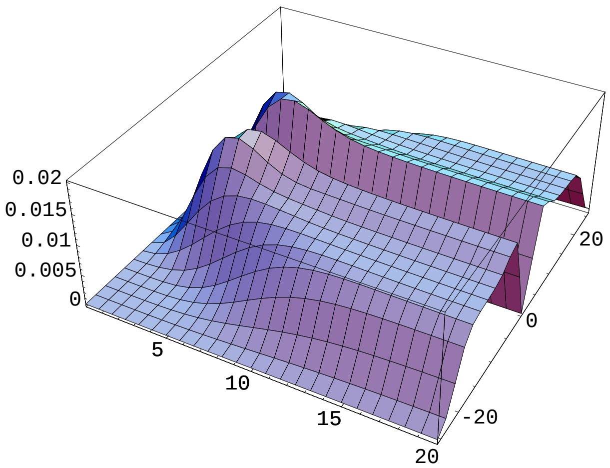
- **cuts** through the complex density for $SU(2)$ staggered [G.A., Bittner, Lombardo, Markum, Pullirsch 04]

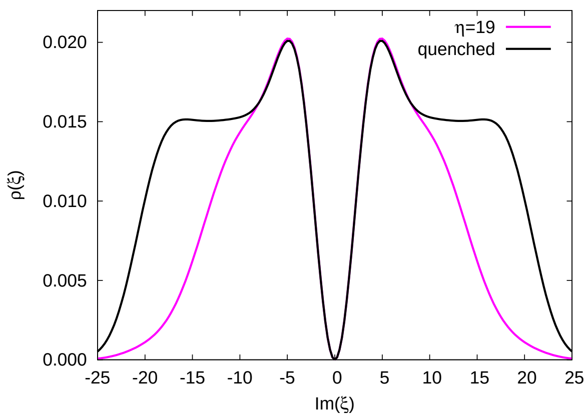
Unquenching SU(2): cut curves



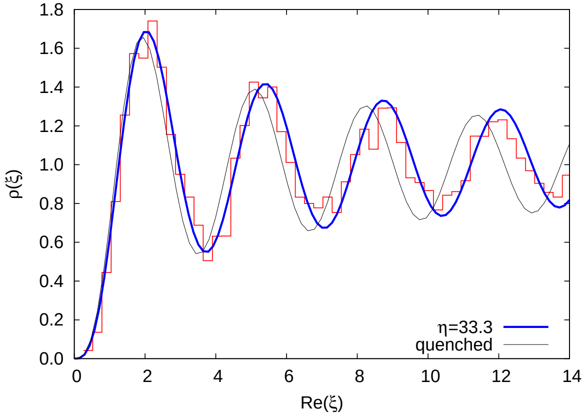


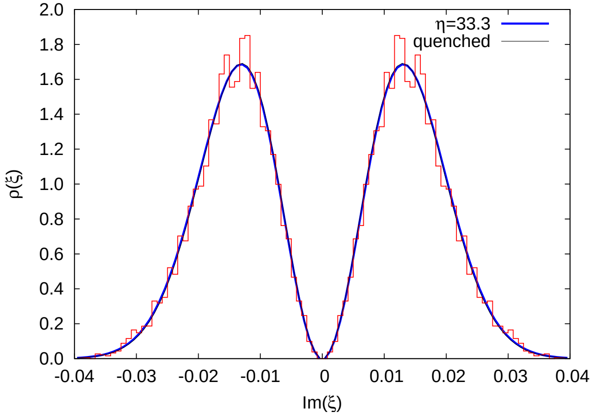
- **shift of $N_f = 2$** at our **mass** parameters



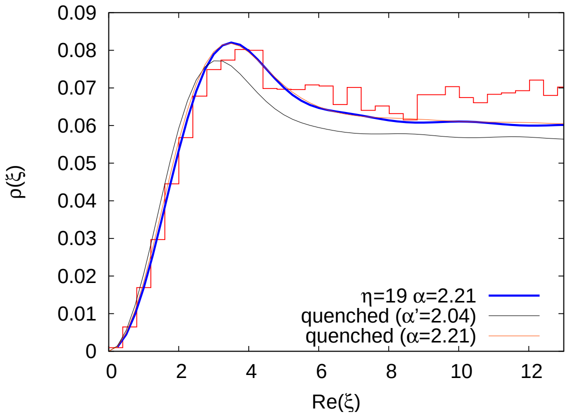


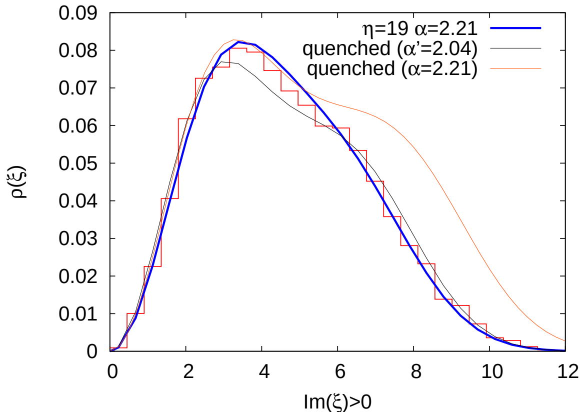
. . . versus unquenched data





- $ma = 0.07$ $V = 6^4$ $\mu = 10^{-3}$ **1 parameter fit**

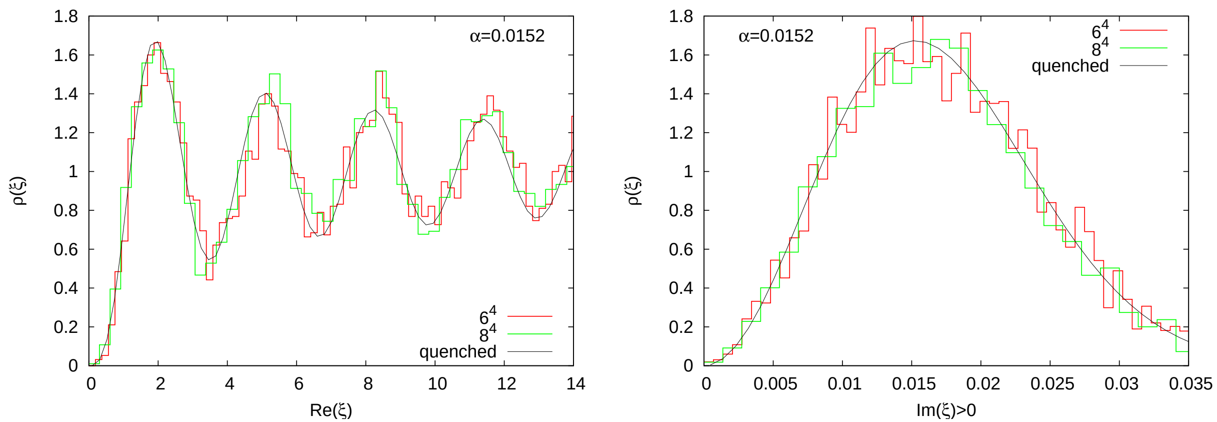




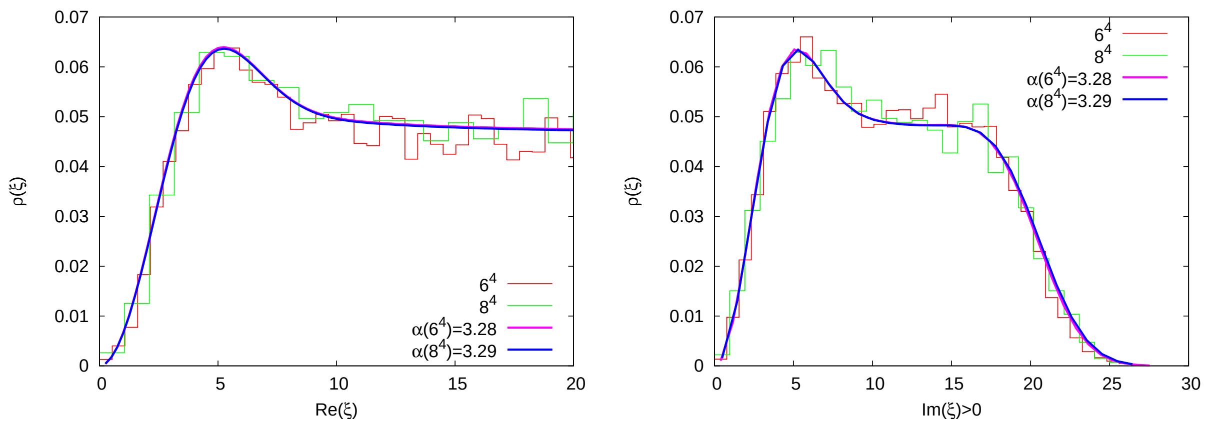
- $ma = 0.06$ $V = 6^4$ $\mu = 0.2$ **1 parameter fit**

(quenched curves for comparison) [G.A., E. Bittner 06]

The scaling $\mu^2 V$=const.



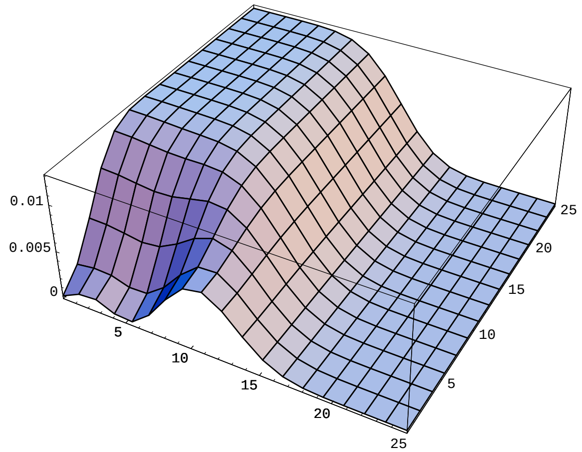
- $\underline{V = 6^4}$ at $\mu = 10^{-3}$ vs. $\underline{V = 8^4}$ $\mu = 5.625 \cdot 10^{-4}$

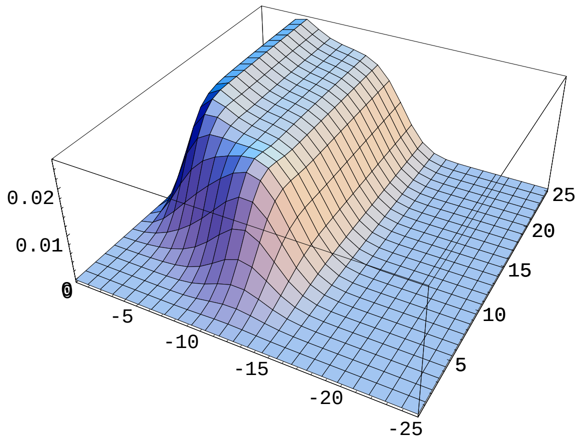


- $\underline{V = 6^4}$ at $\mu = 0.2$ vs. $\underline{V = 8^4}$ $\mu = 0.1125$

Phase quenching vs. unquenching: doable?

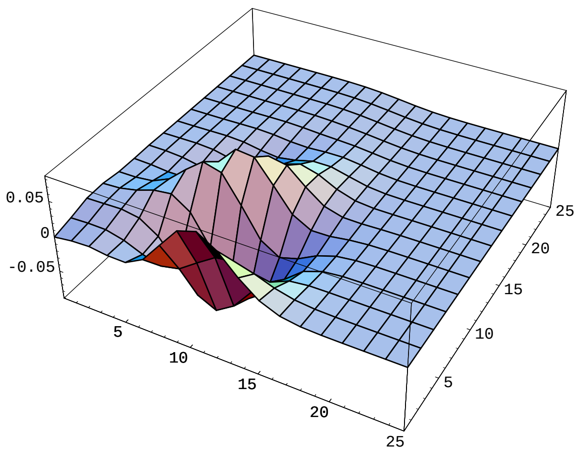
$N_f = 2$, $mV\langle\bar{q}q\rangle = 5$, $\mu^2 f_\pi^2 V = 6.3$: **phase quenched QCD, SU(2)**



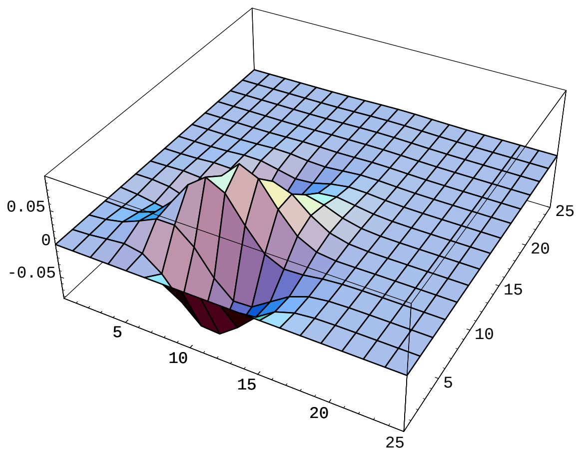


vs. unquenched QCD [G.A.,Osborn,Splittorff,Verbaarschot05; GA05]

$\Re\mathrm{e}(\rho_{MM}(\xi))$

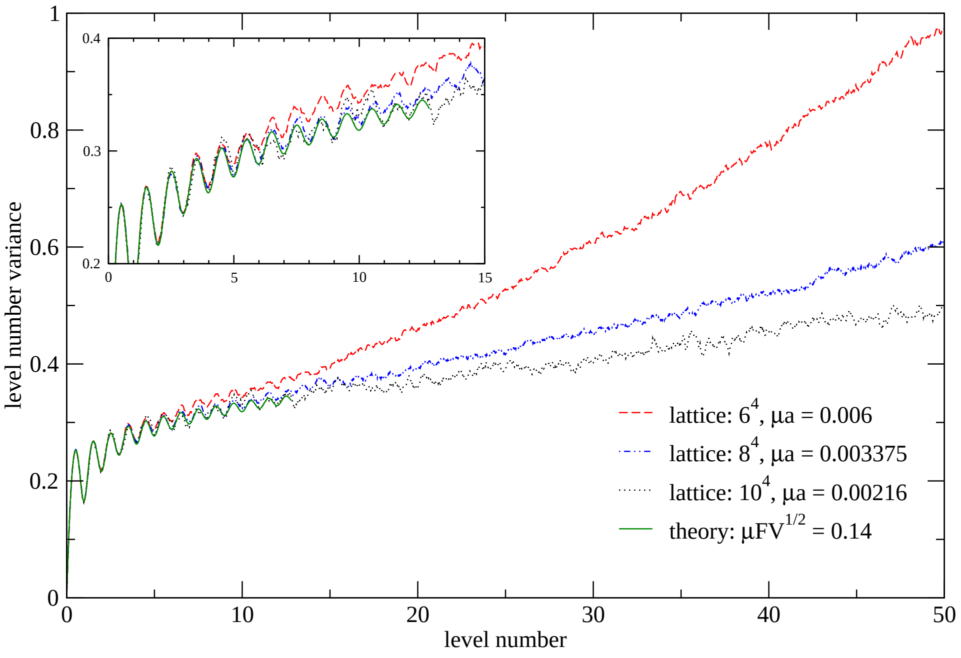


$\Im\mathrm{m}(\rho_{MM}(\xi))$



Leaving the ε-regime

• propagating modes ξ reappear at scale
Thouless energy: $E_{Th} \sim f_\pi/\sqrt{V}$
[Osborn, Verbaarschot 98]



$\mu \neq 0$ [J. C. Osborn, Tilo Wettig, PoS LAT2005 (2005) 200 hep-lat/0510115]

6. Conclusions

- Complex MM have provided many insights and quantitative tests for the Lattice.

open problems:

- 3rd symmetry class: **real non-symmetric matrices**
  Lattice easy vs. **MM hard**

- **relation to χPT + μ in ε-regime: non-QCD classes**

- **unquenched QCD + μ_B on Lattice**