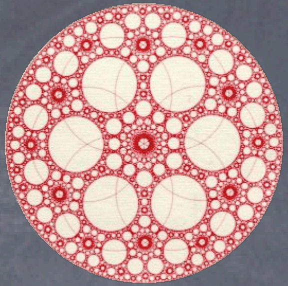

Weak chaos: non perturbative techniques

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Como

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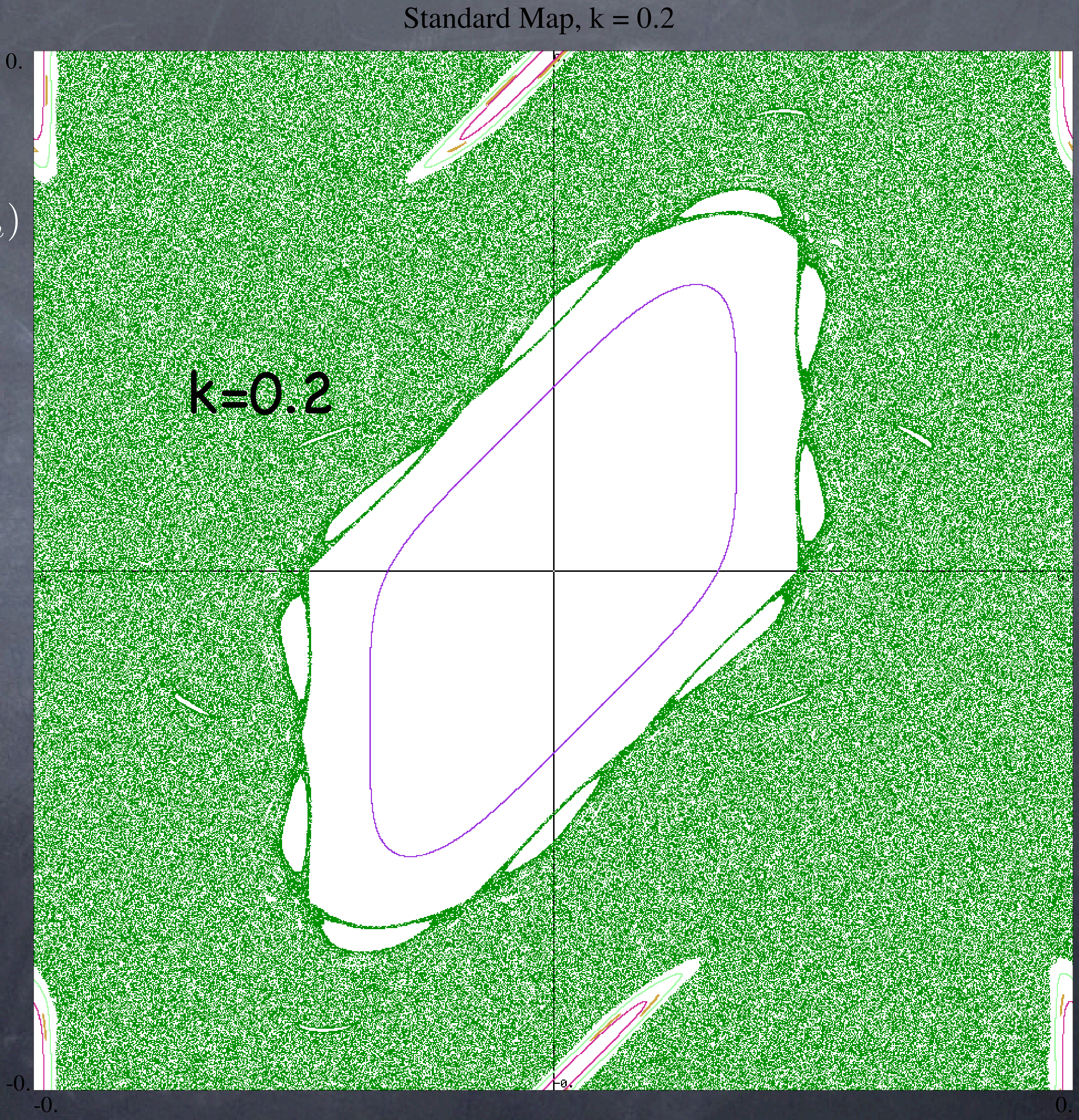
Predrag Cvitanovic'
Per Dahlqvist
Gregor Tanner

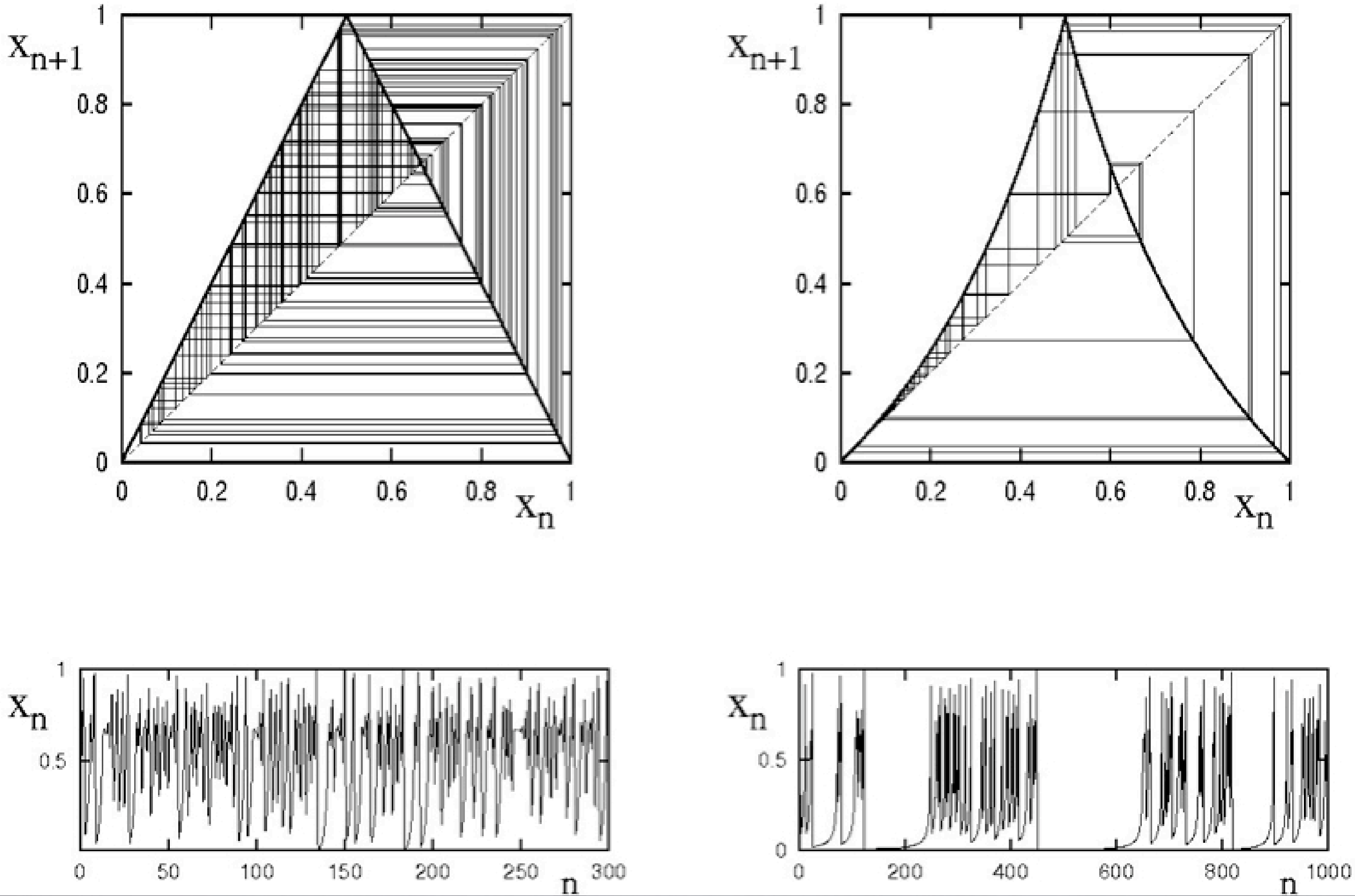
Bari 20-22 September 2006

$$x_{n+1} = x_n + p_{n+1}$$
$$p_{n+1} = p_n - \frac{k}{2\pi}\sin(2\pi x_n)$$

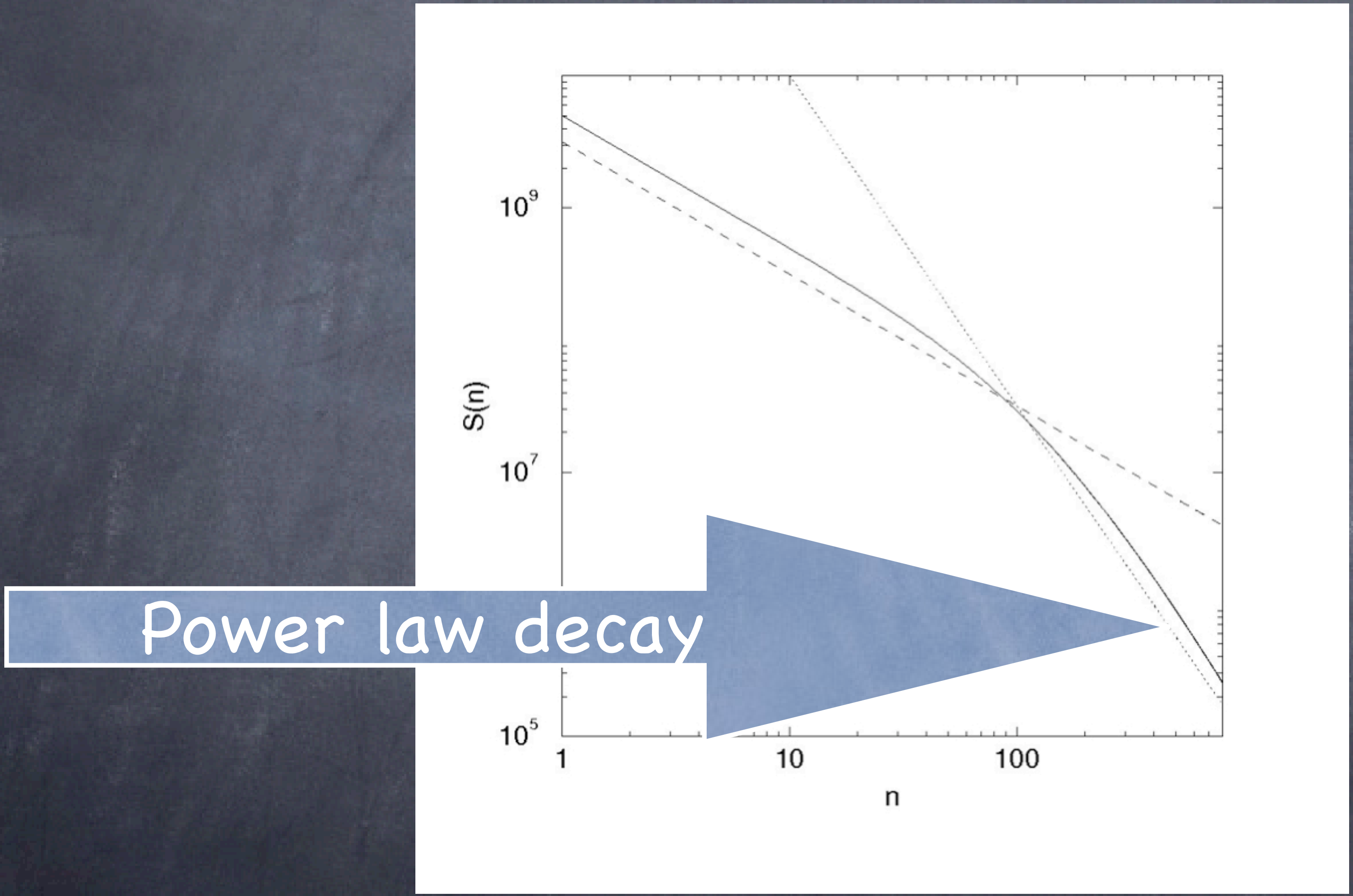


StdMap
J.D.Meiss

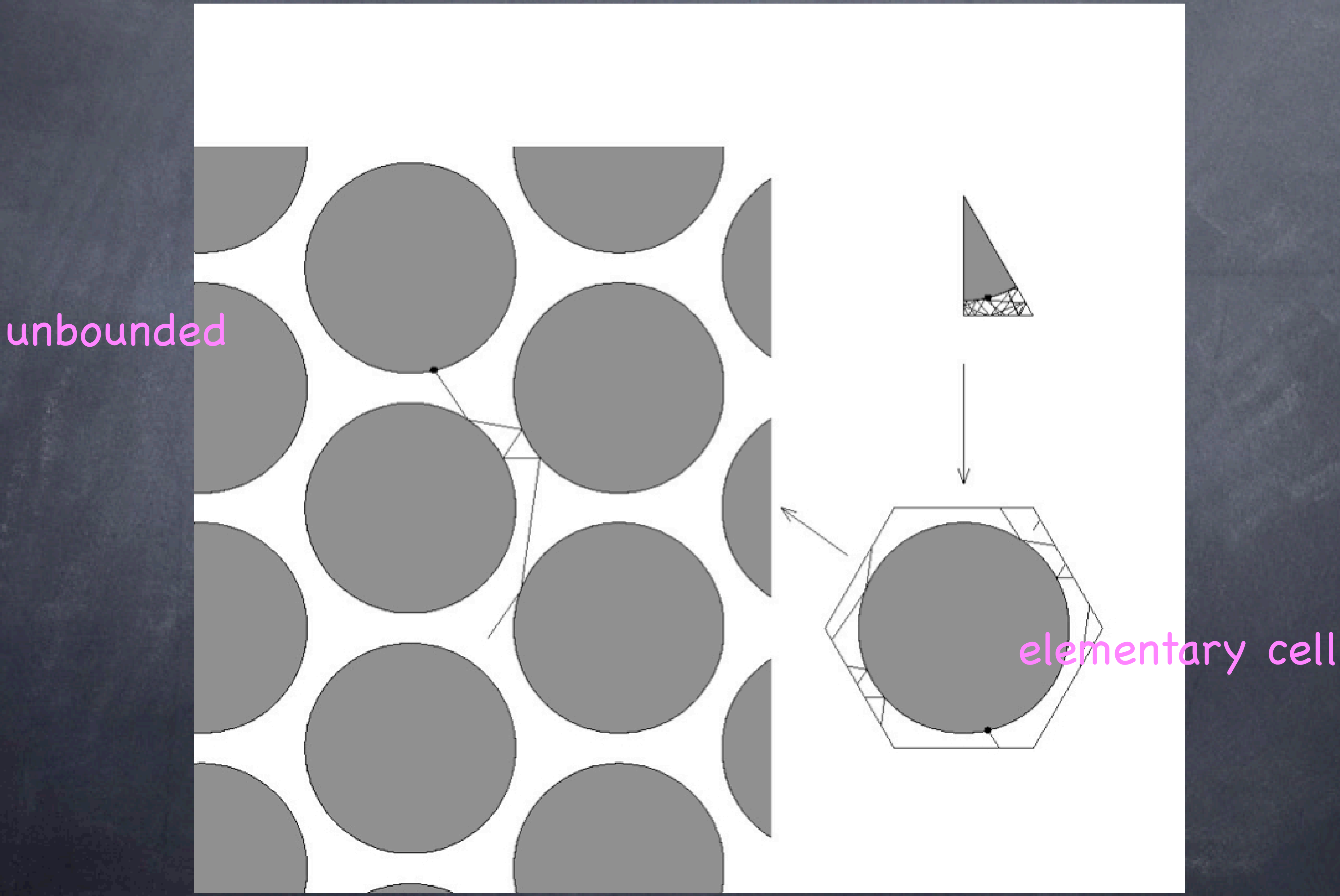
Qualitative intermittency



Sticking to regular islands creates dynamical anomalies



Transport in space periodic deterministic systems

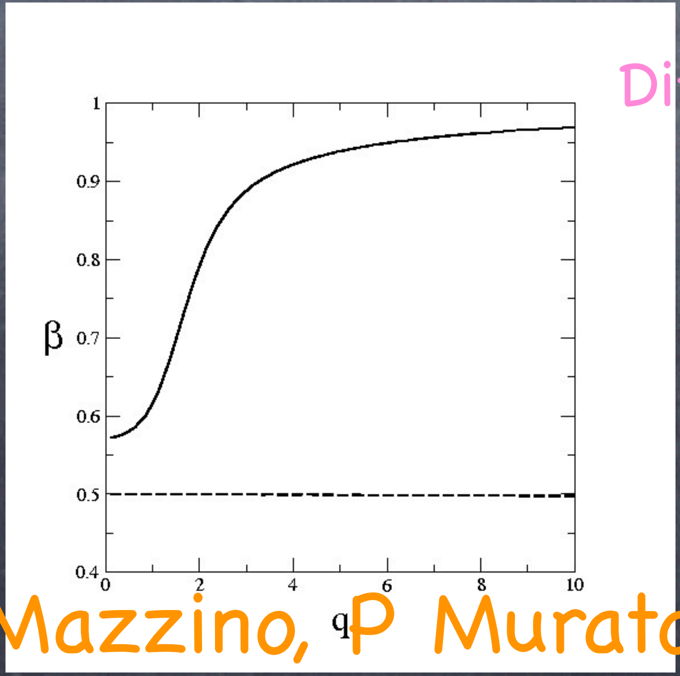


Transport exponents

Anomalous behavior in moments' spectrum

$$\langle | x_t - x_0 |^q \rangle \sim t^{\nu(q)} = t^{q \cdot \beta(q)}$$

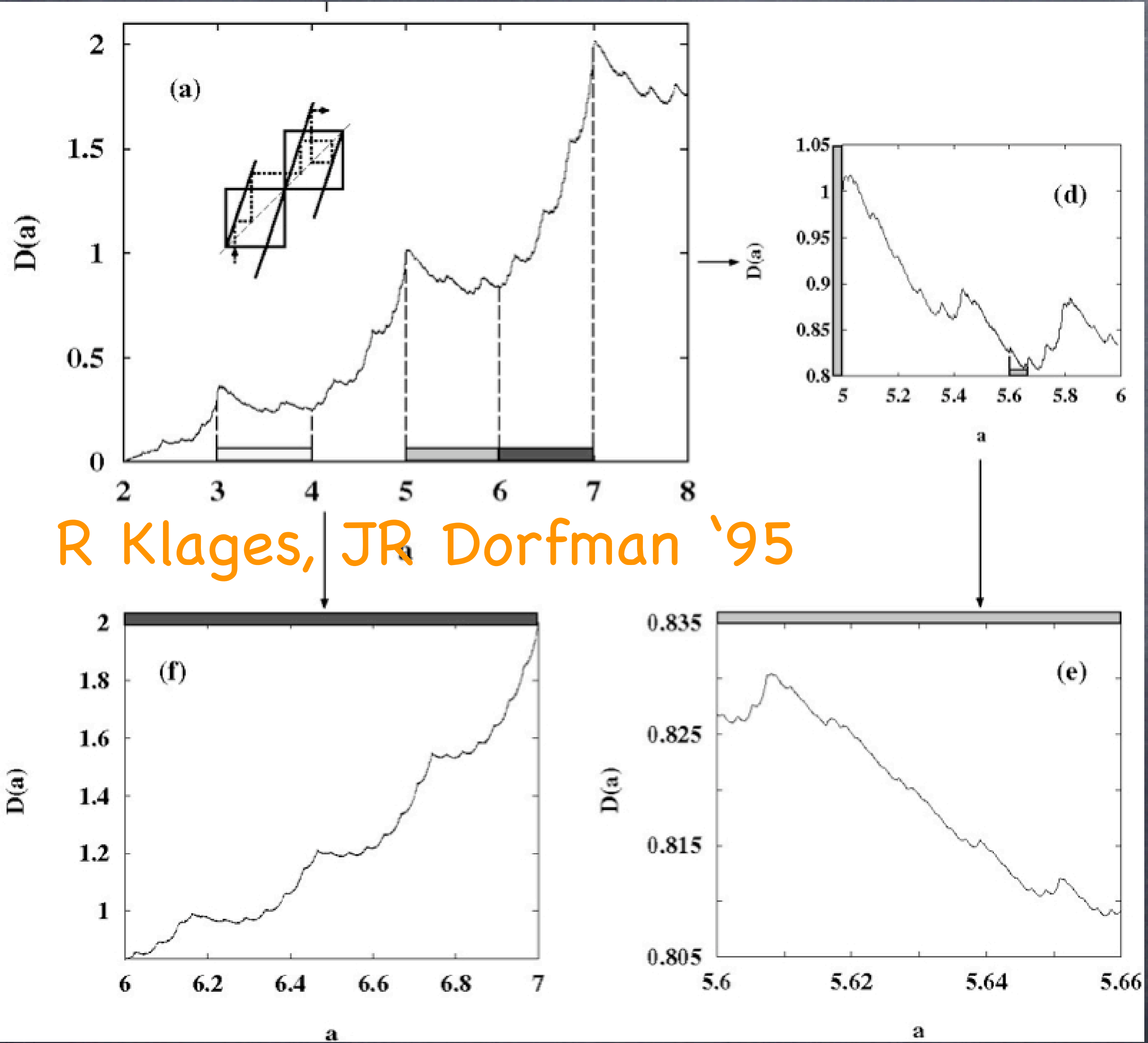
Normal, gaussian transport yields β(q)=1/2



Different parameter values for the standard map

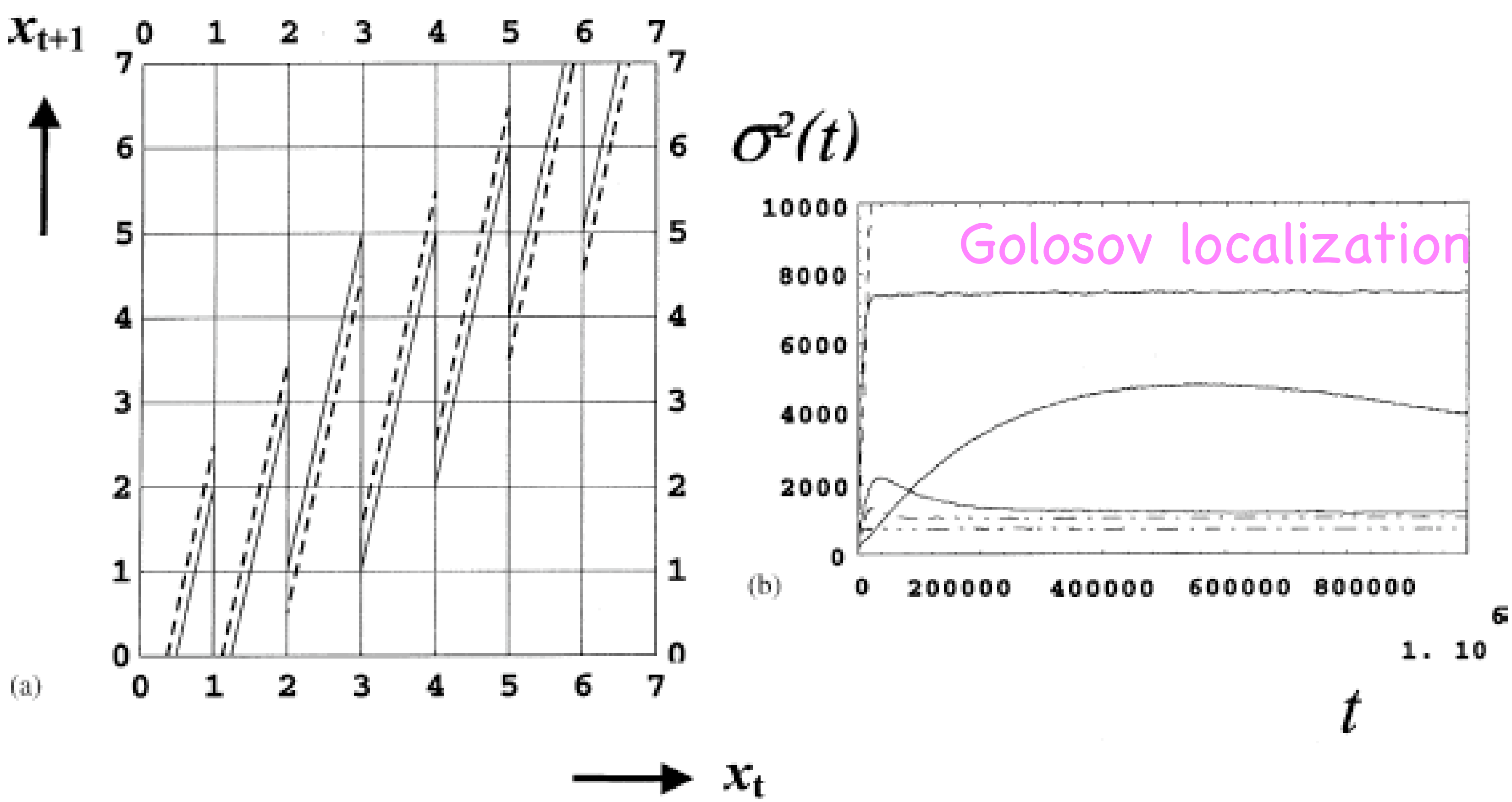
P Castiglione, A Mazzino, P Muratore, A Vulpiani, '99

Even normal is subtle



R Klages, JR Dorfman '95

G Radons, quenched disorder '04



DP Sanders, H Larralde '05

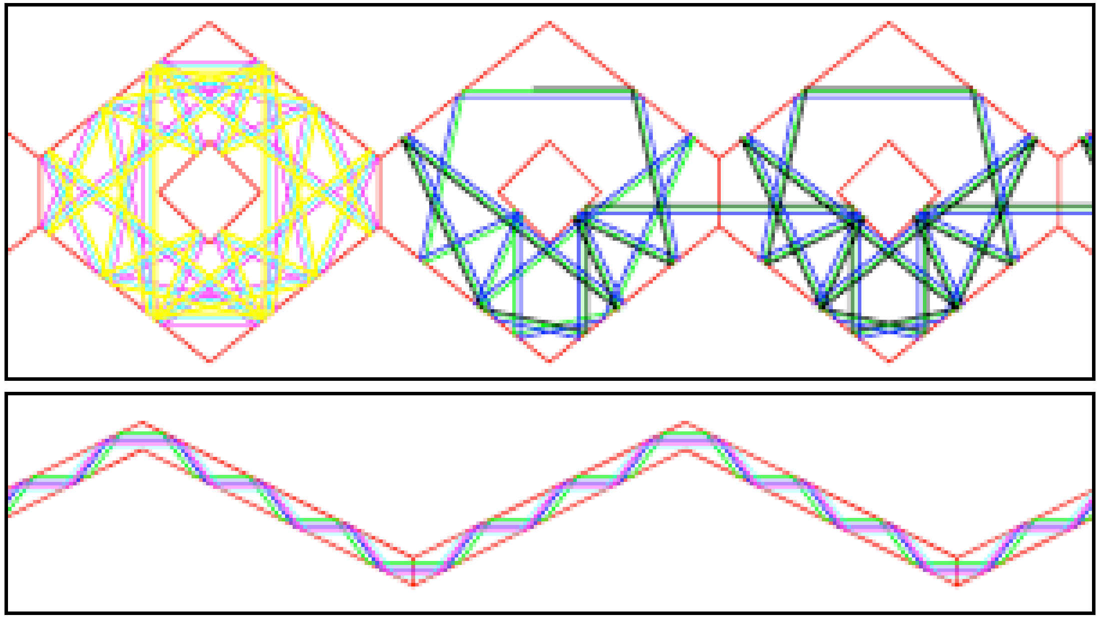
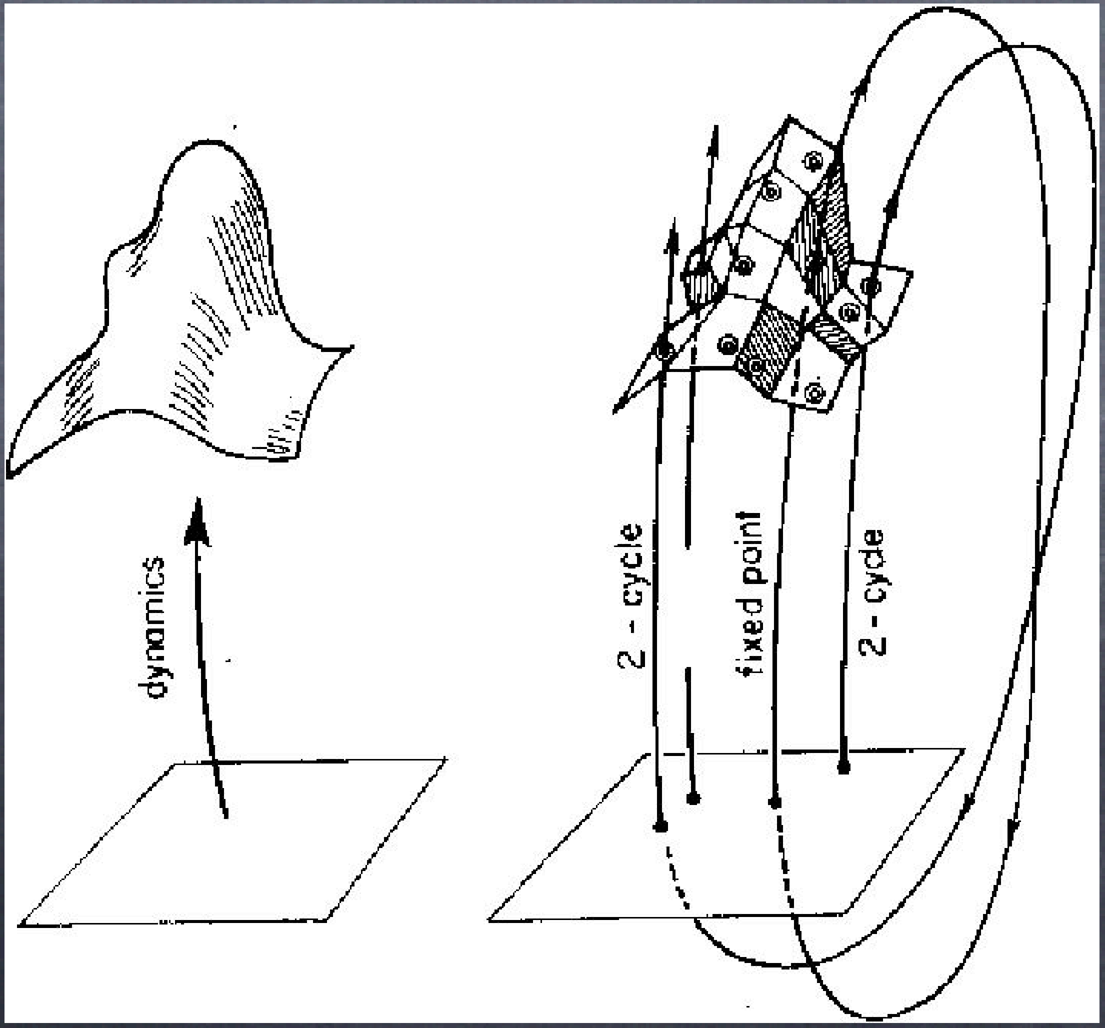


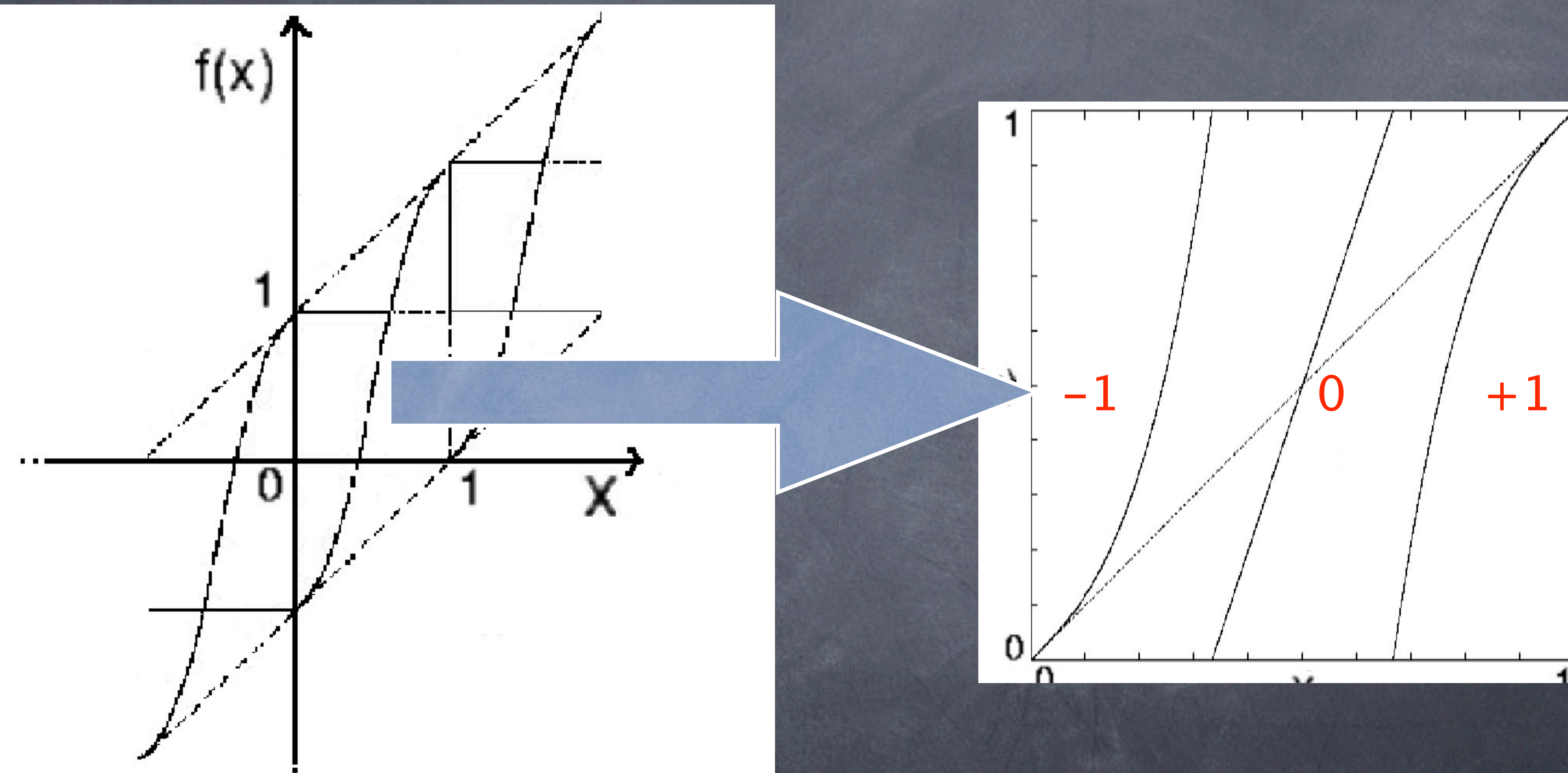
FIG. 13: (Color online) Families of trapped and propagating periodic orbits in parallel systems.

The approach

- Transfer matrix - Perron Frobenius operator
- Employ periodic orbits (families of them)



Unbounded vs torus dynamics



Correspondence is complete once we assign "jumping numbers" σ

Transport in the lift

If we code dynamics according to periodic orbits of the torus map, we have two possible transport outcomes

$$\hat{f}^n(x_i) = x_i \qquad f^n(x_i) = x_i$$

Standing periodic orbit -- no transport

$$\hat{f}^n(x_i) = x_i \qquad f^n(x_i) = x_i + \sigma_i \qquad \sigma_i \in \mathrm{N}$$

Running periodic orbit -- ballistic transport

Transfer operator

We consider the generating function

$$G_t(\beta) = \langle \mathrm{e}^{\beta(\mathrm{x_t} - \mathrm{x_0})} \rangle$$

and the generalized transfer operator

$$[\mathcal{L}_\beta \phi](x) = \int_X \mathrm{dy}\ \mathrm{e}^{\beta(\mathrm{f(y)} - \mathrm{y})} \delta(\mathrm{f(y)} - \mathrm{x}) \phi(\mathrm{y})$$

This is a modified, weighted, Perron Frobenius operator, that maintains semigroup property

Moments' spectrum and eigenvalues

Asymptotic behavior of the generating function is ruled by the leading eigenvalue

$$G_t(\beta) = \int \mathrm{dx}[\mathcal{L}^{\mathrm{t}}_{\beta}\rho_{\mathrm{in}}](\mathrm{x}) \sim \lambda_0^t(\beta)$$

The generating function yields moments' spectrum

$$\langle (x_t - x_0)^q \rangle = \left(\frac{\partial^q}{\partial \beta^q} G_t(\beta) \right) \Big|_{\beta=0}$$

Transfer operator and torus map

Transfer operator may be expressed in terms of the torus map

$$[\mathcal{L}_\beta \rho](\phi) = \int_{\mathcal{T}} d\theta e^{\beta(\hat{f}(\theta)-\theta+\sigma(\theta))}\delta\left(\hat{f}(\theta)-\phi\right)\rho(\theta)$$

Smallest zero of the secular equation yields the inverse of the leading eigenvalue

$$\det(1 - z\mathcal{L}_\beta) = F_\beta(z) = 0$$

The role of periodic orbits

The determinant equation can be rewritten in terms of a dynamical zeta function

$$\lambda_0 = z^{-1}(\beta) \qquad \zeta^{-1}_{(0)\beta}(z(\beta)) = 0$$

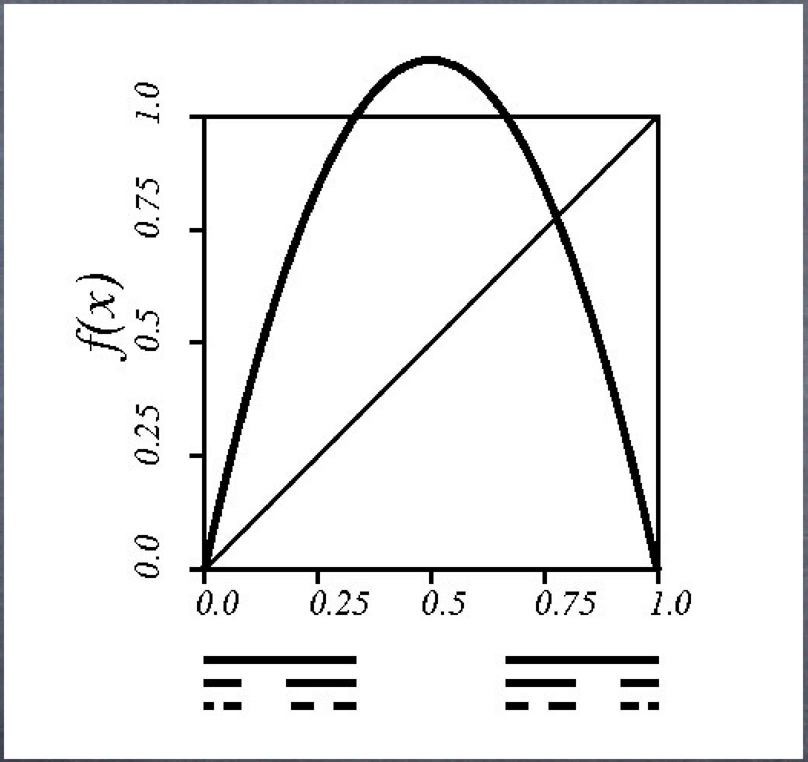
$$\zeta^{-1}_{(0)\beta}(z) = \prod_{\{p\}} \left(1 - \frac{z^{n_p} \mathrm{e}^{\beta\sigma_\mathrm{p}}}{|\Lambda_p|}\right)$$

$$\zeta^{-1}_{(0)\beta}(z) = \prod_{\{p\}} \left(1 - \frac{z^{n_p} \mathrm{e}^{\beta\sigma_\mathrm{p}}}{|\Lambda_p|}\right)$$

- {p} denotes the set of prime periodic orbits for the map on the fundamental cell
- n_p being the period σ_p jumping number and Λ_p the instability

Zeta functions as perturbation theory

When symbolic dynamics is “simple”, converting the product to a sum leads to a perturbative scheme



$$\begin{aligned}
\zeta^{-1}_{(k)\beta}(z) &= \prod_{\{p\}}(1-t_p) = \\
&= (1-t_0)\cdot(1-t_1)\cdot(1-t_{01})\cdot(1-t_{001})\cdot(1-t_{011})\cdots \\
&= 1-t_0-t_1+ \\
& \quad -(t_{01}-t_0t_1)-(t_{001}-t_0t_{01})-(t_{110}-t_1t_{01})-\cdots
\end{aligned}$$

Example of a perturbative calculation

Feigenbaum's δ (universal constant for the period doubling route to chaos)

$$\delta = 4.669\,201\,609\,102\,990\,671\,853\,2038\ldots$$

R.A., E Aurell, P Cvitanovic' '90; F Christiansen, P Cvitanovic', HH Rugh '90

The Works of Jerry Iuliano
Ancient Numbers Revealed in Scientific Formulas
Compiled by Joseph E. Mason
Cheops Pyramid Face Angle
The Cheops pyramid face angle . . . 51.85 degrees...can describe both the fine-structure constant . . . aem = 1/137.036 . . . and the Feigenbaum delta constant . . . Fd = 4.669201609...because of the ability for angle transformation that the fine-structure constant does on the face angle: transforming the face angle from degrees Sumerian..360 . . . to radians ...Pi form. . . :
aem = fine-structure constant = 1/137.036001 . . .
cosine in radians
Cheops pyramid face angle = 51.849982814 . . . degrees Sumerian
(cos 51.849982814) * (10 ^ 4) = - 1/137.036001 . . . = 1/aem
. . . to arrive at the Feigenbaum delta constant, one must subtract the radian from the Cheops face angle. This is interesting because now the direct connection can be shown as a simple subtraction of the radian...180/pi.. from the face angle, which suggests dimensionlessness.
((radian - 51.849982814) ^ (1/4)) + Pi = Fd = 4.669212726 . . .

Integral formulas for moments

$$\langle (x_n - x_0)^k \rangle_0 \sim \frac{\partial^k}{\partial \beta^k} \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} ds\, \mathrm{e}^{\mathrm{sn}} \frac{\mathrm{d}}{\mathrm{ds}} \ln \left[\zeta_{(0)\beta}^{-1}(\mathrm{e}^{-\mathrm{s}}) \right] \Big|_{\beta=0}$$

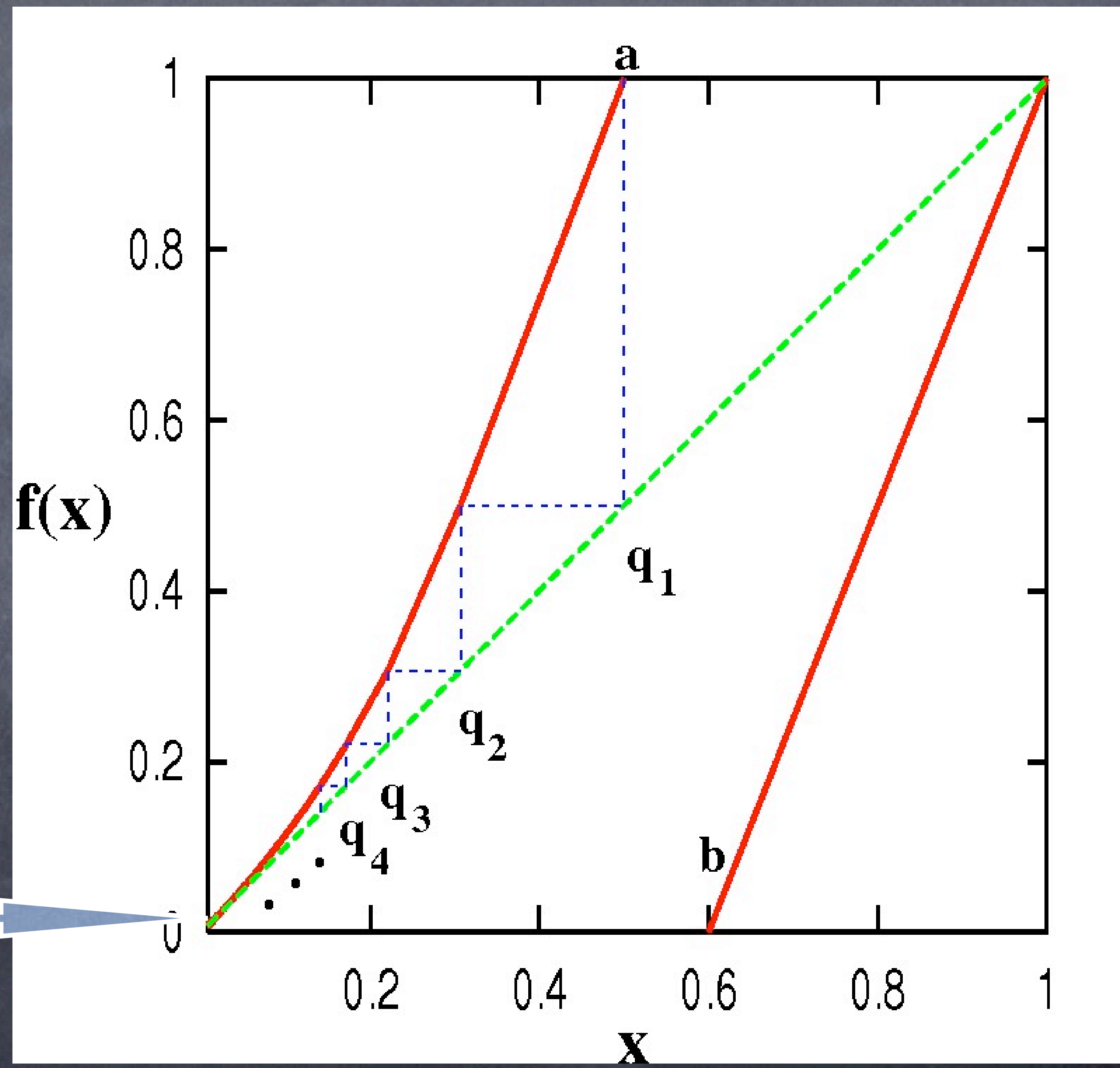
Second order moment yields

$$D = \lim_{n\to\infty} \frac{1}{2n} \frac{\partial^2}{\partial \beta^2} \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} ds\, \mathrm{e}^{\mathrm{sn}} \frac{\mathrm{d}}{\mathrm{ds}} \ln \left[\zeta_{(0)\beta}^{-1}(\mathrm{e}^{-\mathrm{s}}) \right] \Big|_{\beta=0}$$

R.A., ‘91; P Cvitanovic’, JP Eckmann, P Gaspard, ‘95

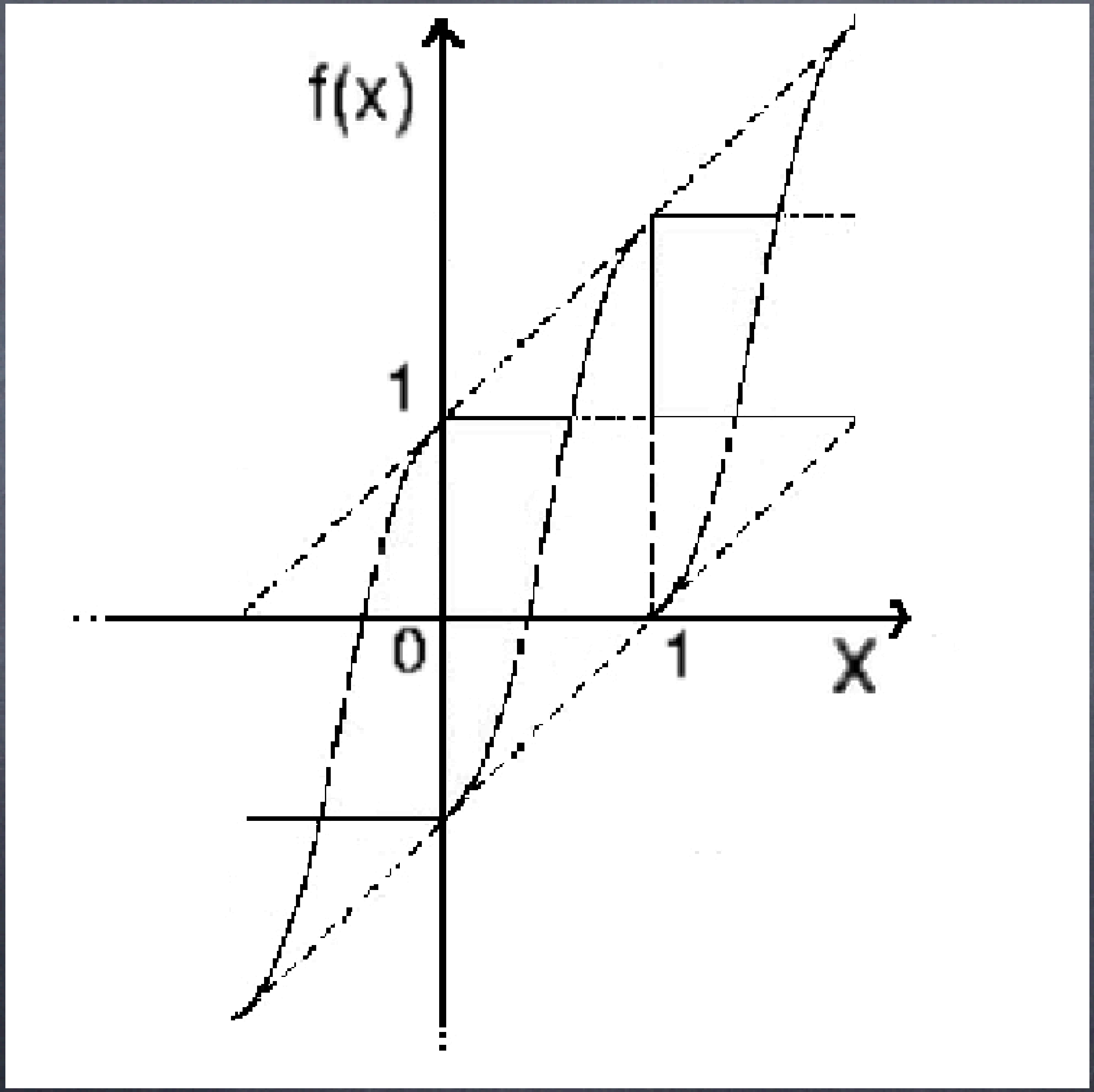
An intermittent paradigm: Pomeau-Manneville

$$x_{n+1} = x_n + x_n^\gamma|_{\mathrm{mod}\ 1}$$



Marginal fixed point

Geisel, Nierwetberg, Zacherl '85



Instability anomaly close to the fixed point

Periodic orbit coming closer and closer to the fixed point exhibit power law instability growth (fully chaotic ones grow exponentially)

$$\Lambda_{0^n 1} \sim n^{\alpha+1}$$

Power law exponent is related to intermittency

$$\alpha = \frac{1}{\gamma - 1}$$

$$x_{n+1} = x_n + x_n^{\gamma}|_{\mathrm{mod}\ 1}$$

Power laws modify analytic properties

Unweighted (Perron-Frobenius) zeta function is written as

$$\zeta^{-1}_{(0)0}(z) = 1 - az - bz\sum_{k=1}^{\infty}\frac{z^k}{k^{\alpha+1}}$$

Compare with a simple chaotic example: uniform instability, simple grammar with a binary alphabet

$$\zeta^{-1}_{(0)0}(z) = 1 - \frac{2z}{\Lambda}$$

non trivial analytic properties

simple zero

Analytic structure

$$\sum_{l=1}^{\infty} \frac{z^l}{l^\mu} = \begin{cases} (1-z)^{\mu-1} & \mu < 1 \\ \ln(1-z) & \mu = 1 \\ \zeta(\mu) + C_\mu (1-z)^{\mu-1} + D_\mu (1-z) & \mu \in (1,2) \\ \zeta(2) + C_2 (1-z) \ln(1-z) & \mu = 2 \\ \zeta(\mu) + C_\mu (1-z) & \mu > 2 \end{cases}$$

Zeta functions (either weighted or unweighted) may display non trivial (i.e. simple zero) behaviors

Tauberian theorem for Laplace transforms

$$\omega(\lambda) = \int_0^\infty dx\, e^{-\lambda x} u(x)$$

$$\omega(\lambda) \sim \frac{1}{\lambda^\rho} L\left(\frac{1}{\rho}\right) \Longleftrightarrow u(x) \sim \frac{1}{\Gamma(\rho)} x^{\rho-1} L(x)$$

$\lambda \to 0$ $\qquad$ $x \to \infty$

Second moment of the distribution

$$\langle (x_n - x_0)^2 \rangle_0 = \frac{\partial^2}{\partial \beta^2} \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} ds\, \mathrm{e}^{\mathrm{sn}} \frac{\mathrm{d}}{\mathrm{ds}} \ln \left[\zeta^{-1}_{(0)\beta}(\mathrm{e}^{-\mathrm{s}}) \right] \Big|_{\beta=0}$$

simple zero -> linear growth with n

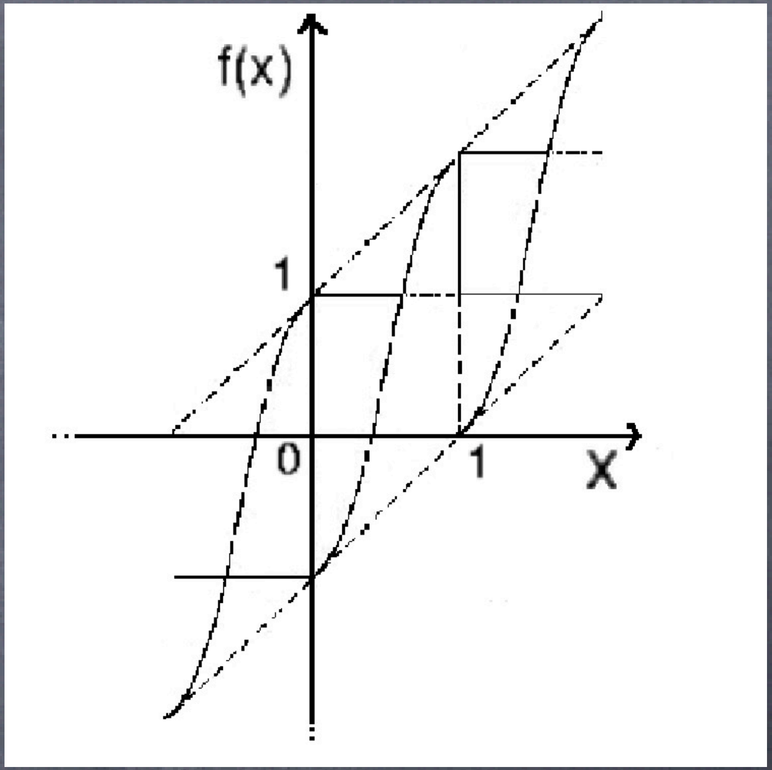
branch cut, logarithmic corrections -> deviations from linear behavior

A brief tour through a simple example

R.A., G Cristadoro, '04

Intermittent "running" fixed points
(ballistic transport)

$$\zeta^{-1}_{(0)\beta}(z) = 1 - az - bz \sum_{k=1}^{\infty} \frac{z^k}{k^{\alpha+1}} \cosh(\beta k)$$



Moments' behavior

$$\langle (x_n - x_0)^k \rangle_0 \sim \frac{\partial^k}{\partial \beta^k} \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} ds\, \mathrm{e}^{\mathrm{s}n} \frac{\mathrm{d}}{\mathrm{ds}} \ln \left[\zeta^{-1}_{(0)\beta}(\mathrm{e}^{-\mathrm{s}}) \right] \Bigg|_{\beta=0}$$

n-th order derivatives are computed by Faá di Bruno formula

$$\frac{d^n}{dt^n} H(L(t)) = \sum_{k=1}^{n} \sum_{k_1, \cdots k_n} \frac{d^k H}{dt^k}(L(t)) \cdot B_{\vec{k}}(L(t))$$

$$B_{\vec{k}}(L(t)) = \left(\frac{1}{1!}\frac{dL}{dt}\right)^{k_1} \cdots \left(\frac{1}{n!}\frac{d^n L}{dt^n}\right)^{k_n} \qquad \vec{k} = \{k_1, \ldots k_n\} \text{ with } \sum k_i = k, \quad \sum i \cdot k_i = n$$

For every n one has to look for the leading singularity: the result depends on α, the exponent ruling power law growth of instabilities of periodic orbits coming closer and closer to the marginal fixed point

In the present example this exponent is written in terms of the intermittency exponent

$$\alpha = \frac{1}{\gamma - 1}$$

Moments spectrum

$\alpha \in (0,1)$ $\quad \nu(q) = q$ $\quad$ ballistic transport

$$\alpha > 1 \qquad \nu(q) = \begin{cases} q/2 & q < 2(\alpha-1) \\ q+1-\alpha & q > 2(\alpha-1) \end{cases}$$

Normal/ballistic "phase transition"

Such a transition is displayed by a remarkable number of systems P Castiglione, A Mazzino, P Muratore, A Vulpiani, '99

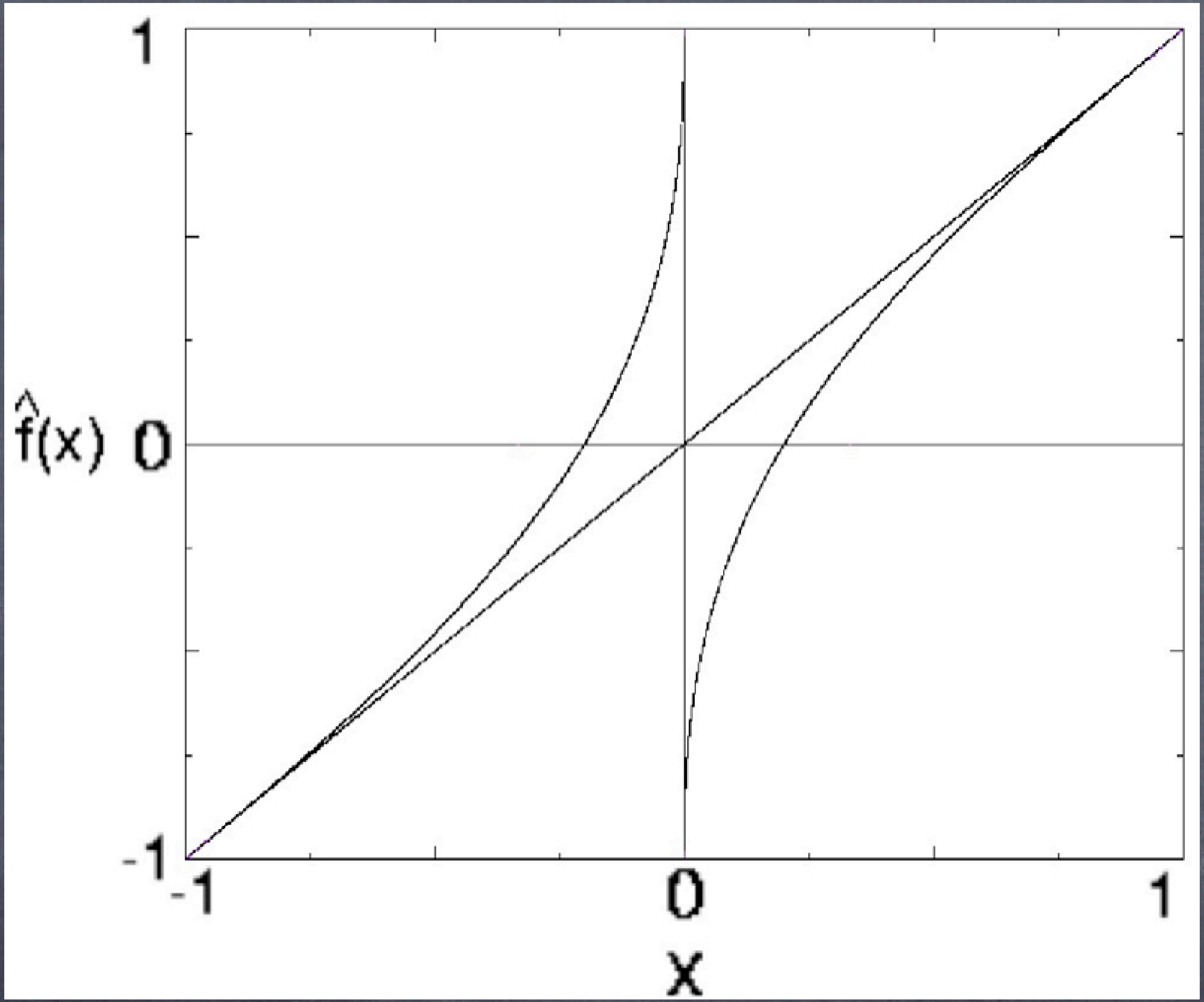
A remark ....

What matters? Pomeau-Manneville maps have invariant measure with singular behavior depending on the intermittency exponent...

$$\begin{cases} x_{t+1} = \hat{f}(x_t) \\ y_{t+1} = y_t + x_t \end{cases}$$

Torus map implicitly defined by

$$x = \begin{cases} \frac{1}{2\gamma}\left(1 + \hat{f}(x)\right)^{\gamma} & 0 < x < 1/(2\gamma) \\ \hat{f}(x) + \frac{1}{2\gamma}\left(1 - \hat{f}(x)\right)^{\gamma} & 1/(2\gamma) < x < 1 \end{cases}$$



Invariant measure is Lebesgue! Former theory still holds: what changes is the link between intermittency exponent and power law instability of cycles approaching the marginal fixed point

Conclusions?

Periodic orbit expansions may be used to characterize the spectrum of anomalous chaotic transport

Exponents ruling the asymptotic moments' growth are tightly related to the power law behavior of periodic orbits coming closer and closer to marginal fixed points

This offers a clue to universality observed for such systems, independently of details about global symbolic dynamics, smoothness properties of invariant measures etc...

Possible future directions

Explore connections with Zaslavsky scaling arguments

Directed transport in intermittent ratchets

How marginality is approached in "generic" area preserving maps

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