Numerical computation of disorder parameters: the case of compact $U(1)$ in four dimensions

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1 – MOTIVATION

Order-disorder transitions are common to a wide class of models in Statistical Mechanics and Quantum Field Theory (Ising model, $3d\ XY$ model, Heisenberg model, ...)

They are usually associated to the spontaneous breaking of a dual symmetry induced by the condensation of dual topological variables (*e.g.* **kinks in the Ising model)**
Correlation functions of the topological variables can be used as disorder parameters for the phase transition.

Dual variables are non local when written in terms of the original variables and their correlation functions are usually expressed as ratios of partition functions
That makes their numerical study quite challenging.

We propose a new method to overcome the problem and apply it to the study of confinement in $4d$ compact U(1) gauge theory

Color confinement is usually believed to be related to the condensation of topological excitations: models can be constructed accordingly, which place the confinement-deconfinement transition into the general scenario of order-disorder transitions.

One appealing model is based on dual superconductivity of the QCD vacuum and relates confinement to the breaking of an abelian dual symmetry induced by the condensation of magnetic monopoles (G. 't Hooft, 1975; S. Mandelstam, 1976; G. Parisi, 1975).

The definition of disorder parameters in that scenario has been studied since long.
In the present work we will apply our method to the parameter $\langle \mu \rangle$ developed by the Pisa group and successfully tested both in pure gauge and full QCD

L. Del Debbio, A. Di Giacomo, G. Paffuti, PLB 349, 513 (1995). A. Di Giacomo, G. Paffuti, PRD 56, 6816 (1997).

A. Di Giacomo, B. Lucini, L. Montesi, G. Paffuti, PRD 61, 034503-034504 (2000)

J. M. Carmona *et al.* PRD 64, 114507 (2001). J. M. Carmona *et al.* PRD 66, 011503 (2002)

M. D'Elia, A. Di Giacomo, B. Lucini, C. Pica, G. Paffuti, PRD 71, 114502 (2005).

Analogous parameters have been studied by different groups

P. Cea, L. Cosmai and A. D. Polosa, PLB 392 (1997) 177; P. Cea and L. Cosmai, JHEP 0111 (2001) 064.

M.N. Chernodub, M.I. Polikarpov, A.I. Veselov, PLB 399, 267 (1997);

J. Frohlich, P.A. Marchetti, PRD 64, 014505 (2001).

2 – DUAL SUPERCONDUCTIVITY IN 4d COMPACT U(1) GAUGE THEORY

The model is defined, in the Wilson formulation, as follows

$$Z(\beta) = \int [d\theta] e^{-\beta S} \; ; \qquad S = \sum_{\vec{x},t,(\mu\nu)} (1 - \cos(\theta_{\mu\nu}(\vec{x},t))$$

$\theta_{\mu\nu}$ **is the phase of the plaquette in the** $\mu - \nu$ **plane starting from the lattice site** $(\vec{x},t)$

It has a critical point at $\beta_c \simeq 1.01$**, which is usually believed to be weak first order and separates a disordered phase (**$\beta < \beta_c$**), with condensation of magnetic monopoles and confinement of electric charges (dual superconductive phase), from a Coulomb phase where condensation of magnetic charge disappears.**

The disorder parameter $\langle\mu\rangle$ is the expectation value of an operator which creates a magnetic monopole by shifting the quantum fields by the monopole vector potential

$$\mu(\vec{y},t_0) = \exp\left[i\frac{1}{e}\int \mathbf{d}^3x\, \vec{E}(\vec{x},t_0)\vec{b}(\vec{x}-\vec{y})\right]$$

it can be discretized on the lattice follows

$$\mu = e^{\beta\sum_{\vec{x},i}(\cos(\theta_{0i}(\vec{x},t_0)-b_i(\vec{x}-\vec{y}))-\cos(\theta_{0i}(\vec{x},t_0)))} \equiv e^{-\beta\Delta S}$$

so that

$$\langle\mu\rangle = \frac{\int(\mathcal{D}U)\, e^{-\beta(S+\Delta S)}}{\int(\mathcal{D}U)\, e^{-\beta S}} = \frac{\int(\mathcal{D}U)\, e^{-\beta\tilde{S}}}{\int(\mathcal{D}U)\, e^{-\beta S}} \equiv \frac{\tilde{Z}(\beta)}{Z(\beta)}\,,$$

$$\tilde{S} = \sum_{\vec{x},t\neq t_0,(\mu\nu)}(1-\cos(\theta_{\mu\nu})) + \sum_{\vec{x},i}(1-\cos(\theta_{0i}(\vec{x},t_0)-b_i(\vec{x}-\vec{y}))) \equiv S+\Delta S$$

The numerical difficulties encountered in the numerical determination of $\langle\mu\rangle$ can be ascribed to the poor overlap between the two statistical distributions in configuration space corresponding to Z and $\tilde{Z}$

μ gets significant contributions only on those configurations having very small statistical weight. The problem increases with the system size as the two distributions shrink towards non overlapping delta functions.

A successful way out is to measure susceptibilities from which $\langle\mu\rangle$ can eventually be reconstructed (L. Del Debbio, A. Di Giacomo, G. Paffuti, PLB 349, 513 (1995). A. Di Giacomo, G. Paffuti, PRD 56, 6816 (1997).)

$$\rho = \frac{d}{d\beta}\ln\langle\mu\rangle = \langle S\rangle_S - \langle\tilde{S}\rangle_{\tilde{S}}\,; \qquad \langle\mu\rangle = e^{\int_0^\beta \rho(\beta')\mathrm{d}\beta'}$$

While that is enough to test $\langle\mu\rangle$ as a parameter for confinement, a direct determination could be useful in contexts like the study of its correlation functions.

3 – THE METHOD

Our proposal is to determine the ratio of partition functions by using intermediate distribution functions having a reasonable overlap with both statistical ensembles corresponding to Z and $\tilde{Z}$. We rewrite the ratio as the product of N distinct ratios:

$$\frac{\tilde{Z}}{Z} = \frac{Z_N}{Z_{N-1}} \frac{Z_{N-1}}{Z_{N-2}} \cdots \frac{Z_1}{Z_0}$$

where $Z_N \equiv \tilde{Z}$, $Z_0 \equiv Z$ and

$$Z_k \equiv \int [d\theta] e^{-\beta S_k} \; ; \qquad S_k \equiv \frac{N-k}{N} S + \frac{k}{N} \tilde{S} \, .$$

The idea is to compute each single ratio by a different Monte Carlo simulation: the difficulty of dealing with N simulations should be greatly compensated by the increased overlap in the distributions corresponding to each couple of partition functions.

Boundary conditions are chosen in a consistent equal way for all partition functions

As a second step to further improve the overlap, we compute each single ratio by using an intermediate distribution

$$\frac{Z_{k+1}}{Z_k} = \frac{\langle \exp\left(-\beta \Delta S/2N\right) \rangle_{k+1/2}}{\langle \exp\left(\beta \Delta S/2N\right) \rangle_{k+1/2}}$$

where each expectation value is computed with the action

$$S_{k+1/2} \equiv (1 - (k+1/2)/N)\ S + ((k+1/2)/N)\ \tilde{S}$$

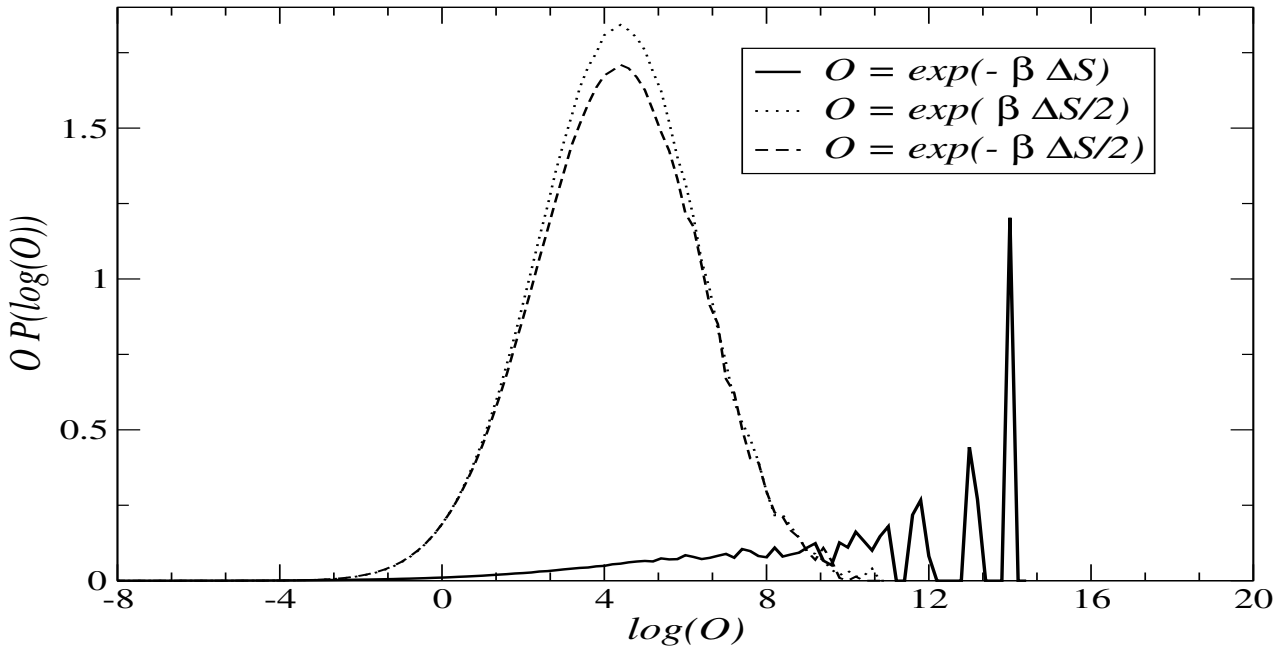
Our method of rewriting the original ratio $\tilde{Z}/Z$ as a product of intermediate ratio resembles very closely the snake algorithm developed for the computation of the 't Hooft loop (Ph. de Forcrand, M. D'Elia, M. Pepe, PRL 86, 1438 (2001)), but differs from it in the choice of the intermediate partition functions.

Our choice is not particularly related to the model or topological excitations taken into account, so that it can be applied in a straightforward way to a wider class of similar problems.

4 – NUMERICAL TESTS

We compare the naïve computation of $\langle \mu \rangle$ with our method for $N = 1$:

- **$\beta = 0.8$, 4^4 lattice, 10^7 measurements for both cases**
- **$\langle \mu \rangle = 1.14(18)$ with the naïve computation and $\langle \mu \rangle = 0.868(3)$ with our method**

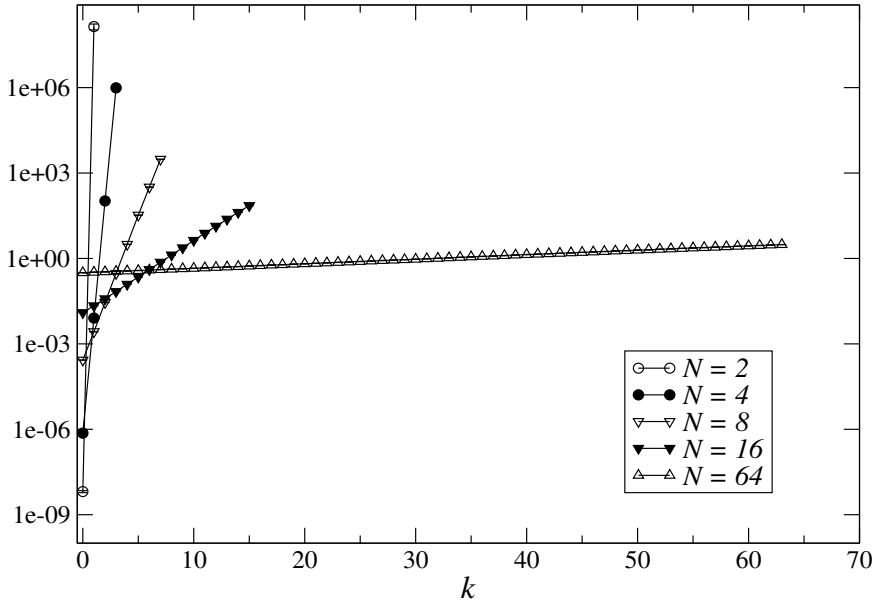


Distributions of the observables $\exp(-\beta \Delta S)_S$ and $\exp(\pm \beta \Delta S/2)_{(S+\tilde{S})/2}$

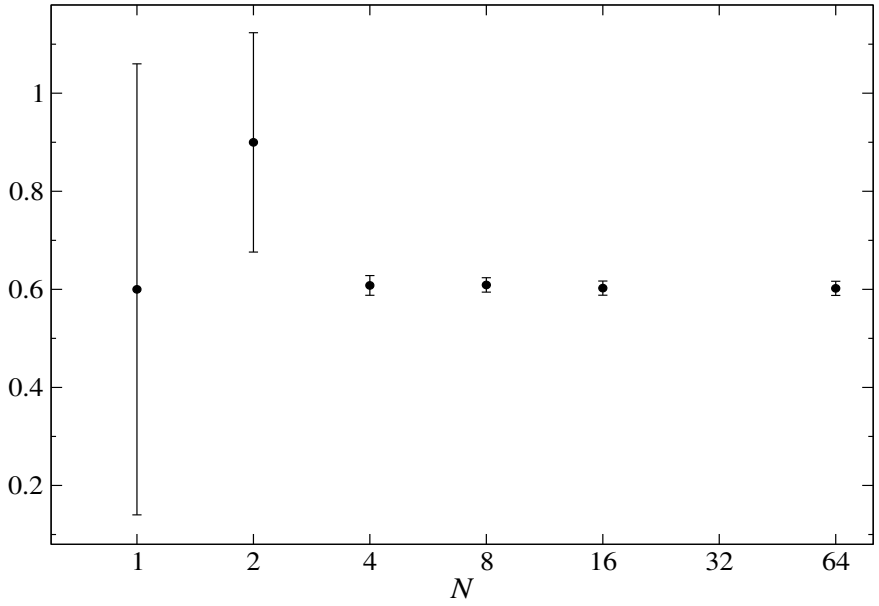
In the naïve computation most of the signal comes from regions which are badly sampled: on larger lattices that makes the determination unfeasible.

We show in the following how results depend on the choice of N

$\langle\mu\rangle$ has been determined on a lattice 16^4 with free b.c. at $\beta = 0.8$: for each determination a whole statistics of $N \times N_{\rm meas} = 3.2 \cdot 10^5$ measurements has been used.



Intermediate ratios as a function of k



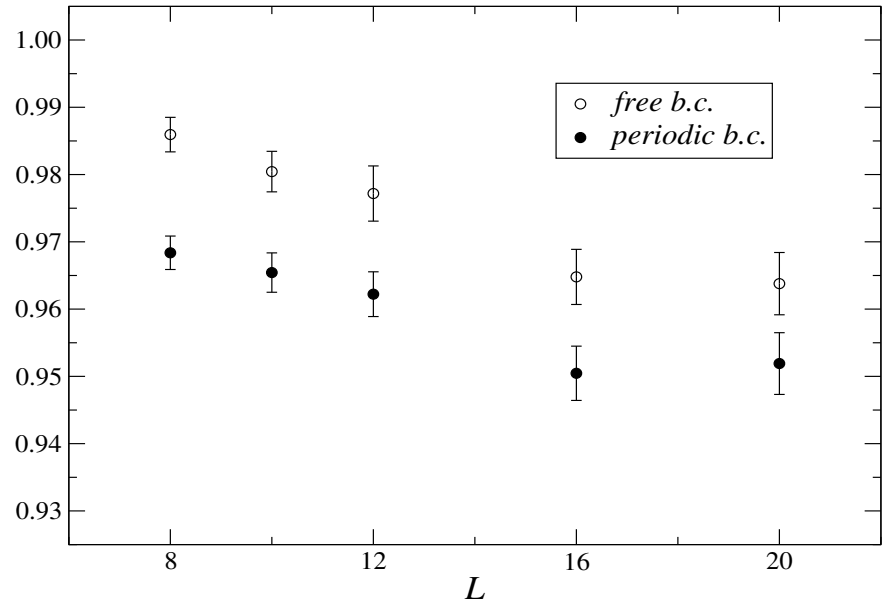
$\langle\mu\rangle$ as a function of N

intermediate ratios are strongly dependent on N whilst $\langle\mu\rangle$ is not, thus confirming the absence of uncontrolled systematic errors.

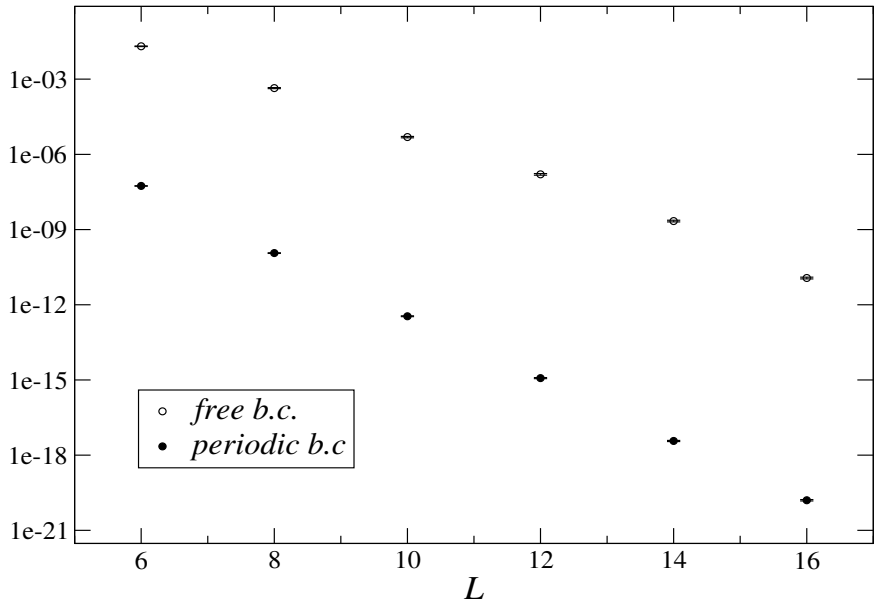
The statistical error rapidly changes for small values of N, but then stabilizes, indicating that a value $N \sim O(10)$ saturates the improvement.

5 – NUMERICAL APPLICATIONS

We have studied the thermodynamical limit of $\langle \mu \rangle$ with different boundary conditions



Size dependence of $\langle \mu \rangle$ at $\beta = 0.5$



Size dependence of $\langle \mu \rangle$ at $\beta = 1.1$

In the confined phase, a fit according to $\langle \mu \rangle = A + B/L$ gives $A = 0.945(6)$ and $B = 0.31(6)$ with free b.c. ($\tilde{\chi}^2 = 0.5$) and $A = 0.940(6)$ and $B = 0.2(6)$ with periodic b.c. ($\tilde{\chi}^2 = 0.8$): $\langle \mu \rangle$ has a well defined thermodynamical limit which is independent of the boundary conditions chosen.

In the deconfined phase $\langle \mu \rangle$ goes to zero exponentially with L, in agreement with magnetic charge superselection and with the behavior of the susceptibility ρ.

We have also performed, in the confined phase, checks of the expected cluster property for the correlation functions of the disorder parameter

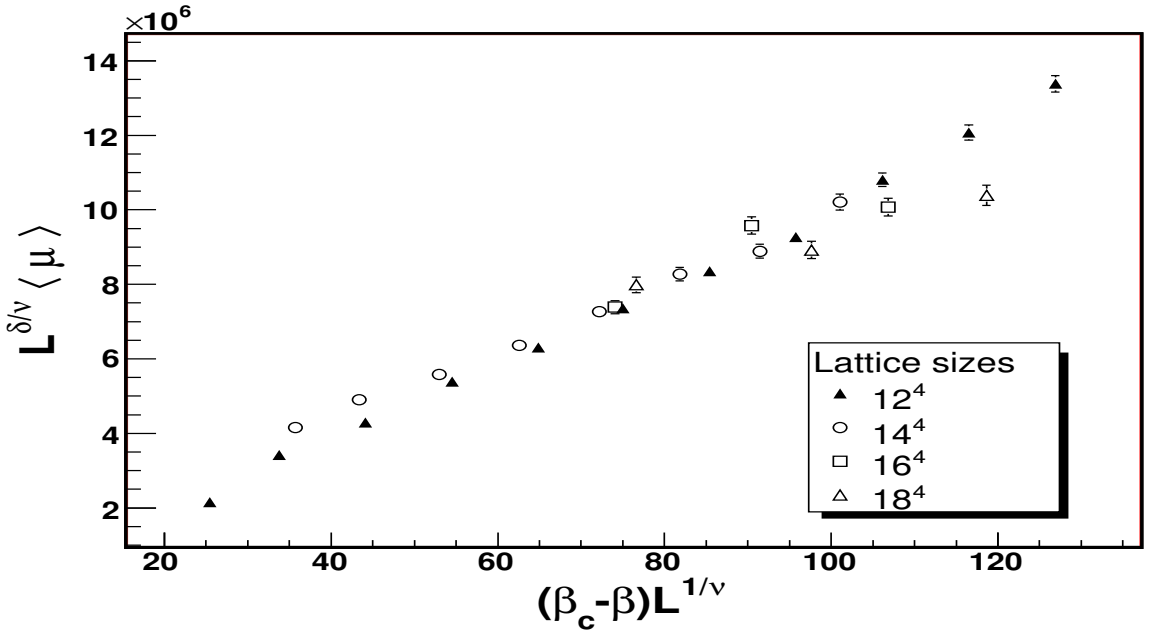
	$\mathcal{O}$	$\mathcal{O}^2$	$\mathcal{O}^4$
$\langle\mu(\vec{0},0)\rangle$	**0.439(12)**	**0.193(11)**	**0.037(4)**
$\langle\bar{\mu}(\vec{0},t)\mu(\vec{0},0)\rangle$	**0.182(7)**	**0.033(3)**	
$\langle\bar{\mu}(\vec{z},0)\mu(\vec{0},0)\rangle$	**0.183(12)**	**0.033(4)**	
$\langle\bar{\mu}(\vec{z},0)\bar{\mu}(\vec{0},t)\mu(\vec{z},t)\mu(\vec{0},0)\rangle$	**0.037(6)**		

We show in the table a determination at $\beta = 0.8$ on a 16^4 lattice of $\langle\mu\rangle$, of its spatial and temporal 2-point function (second and third row) and of its mixed 4-point function (last row), with $t = 8$ and $\vec{z} = (0,0,8)$. All measurements are compatible with the hypothesis that the correlators are already in their asymptotic regime governed by cluster property.

Finite Size Scaling analysis around the phase transition

The critical behaviour of the disorder parameter expected at the phase transition, $\langle \mu \rangle \simeq \tau^{\delta}$, translates in the following ansätz for the f.s.s. behaviour of $\langle \mu \rangle$

$$\langle \mu \rangle = L^{-\frac{\delta}{\nu}} \phi \left((\beta_c - \beta) L^{1/\nu} \right)$$



Fixing the known value of $\beta_c = 1.011$ and $\nu = 1/d$ as appropriate for a weak first order transition, we obtain a reasonable scaling with $\delta \sim 2.3$

6 – CONCLUSIONS AND PERSPECTIVES

- We have proposed a new method for the computation of disorder parameters and applied it to the parameter for dual superconductivity in $4d$ compact U(1) gauge theory

- We have tested our method and used it to study some relevant properties of the disorder parameter $\langle \mu \rangle$ and of its correlation functions

- Our method is very general and its formulation permits both an easy adaption to several similar problems (like the study of order-disorder transitions in statistical models and of dual superconductivity in non Abelian gauge theories) and the possibility to easily implement further improvements (like a different choice of the interpolating actions S_k, for instance not linear in k).