# A closer look at Critical Points

## Slow dynamics, aging and their universal features

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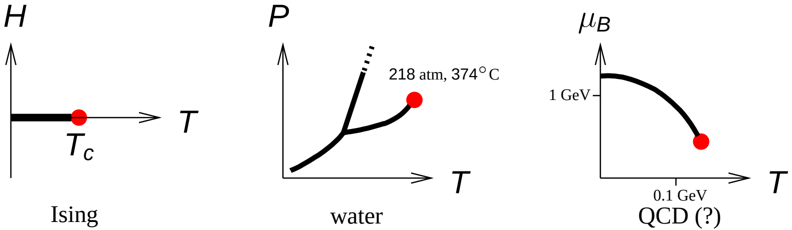
# Critical Points & Critical Dynamics (reminder)




Order parameter: $\phi_x(t)$

Correlated fluctuations: $\xi$, $T_R$

Hallmarks: ST $\xi \gg r_{\mathrm{micr}}$ *collective*

DY $T_R \gg \tau_{\mathrm{micr}}$ *slow*

$\Longrightarrow$ *Universality:* » experimental fact: $\chi \sim a\, \xi^{-\gamma/\nu}$

» minimal models (effective theories, FT, RG and all that)

# Critical Dynamics

TH:

late '70s: Hohenberg, Halperin...
basic minimal models

*...theoretical work...*

2005: Folk, Moser (review)

EX:

'70-'80s: neutron scattering
$S(\mathbf{q}, \omega)$, magn.

*...exp. difficulties...*

2005: X-ray[a]
"watching" critical relax.

[a] Mocuta et al. Science **308**, 1287

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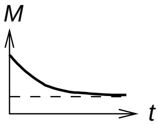
Mostly on: (a) Equilibrium DY close to CP

» $S(\mathbf{q}, \omega)$

» *linear* response $\kappa$, $\eta$, etc. $T_R^{(l)} \sim \xi^z$

(b) Non-linear relaxation

» $M(t) = \langle \phi_x(t) \rangle$, $T_R^{(nl)} \sim \xi^{z-\beta/\nu}$



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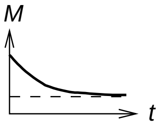
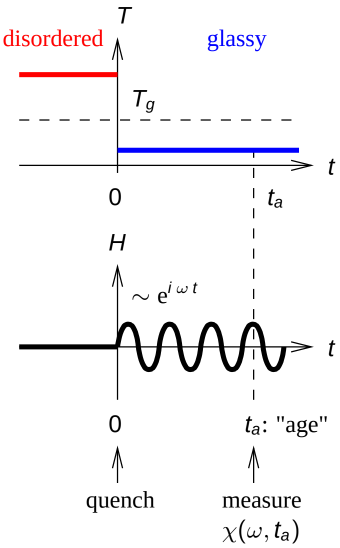
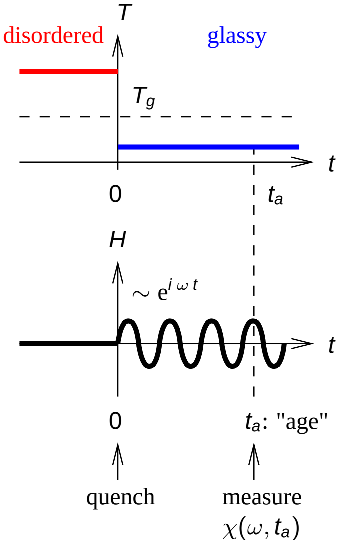



Two-time quantities display *aging*! [CKP'94]



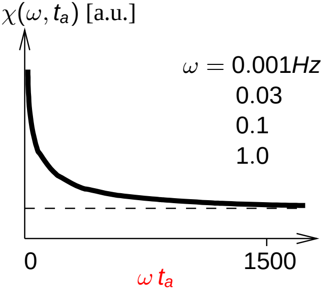
T
disordered
glassy
$T_g$
t
0
$t_a$
H
$\sim e^{i\omega t}$
t
0
$t_a$: "age"
quench
measure
$\chi(\omega, t_a)$

# Aging (spin glasses)




$CdCr_{1.7}In_{0.3}S_4$

$T = 0.72 T_g$



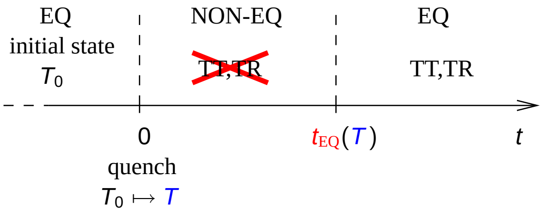

$T_R \sim t_a$

*"the older, the slower"*

[see Vincent et al, cond-mat/9607224]

# Relaxation

Ferromagnet $[T_c, \phi_x(t)]$ relaxing towards *equilibrium*



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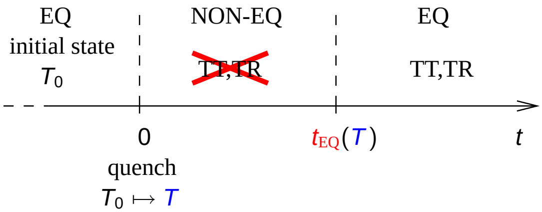



$T > T_c$ : $t_{\rm EQ} < \infty \Longrightarrow$ EQ

$T < T_c$ : $t_{\rm EQ} = \infty \Longrightarrow$ phase ordering dynamics

$\boxed{T = T_c}$ : $t_{\rm EQ} = \infty \Longrightarrow$ Critical Dynamics

$$\xi(t) \sim t^{1/z}$$

# Dynamic Observables I

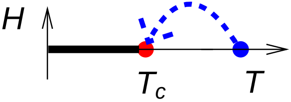
$\phi_x(t)$ order param. (eg, *local* fluct. magn.)

Simplest obs.: (1) one-time quantities $M$
(2) two- $R$, $C$

$M(t) \equiv V^{-1} \sum_x \langle \phi_x(t) \rangle$ *global* $[= \langle \phi_{q=0}(t) \rangle]$

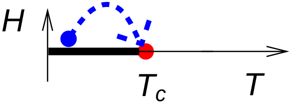
$\boxed{T = T_c}$ :

$M(0) = 0$




$\Longrightarrow M(t) = 0, \quad \forall t$

$M(0) \neq 0$




$\Longrightarrow M(t \to \infty) \sim t^{-\beta/\nu z}$

$M(t \to \infty) = 0 \Longrightarrow$ no infos on dynamics!

# Dynamic Observables II

$$s < t \qquad \begin{cases} C_{x-y}(t,s) \equiv \langle \phi_x(t)\phi_y(s)\rangle_{\text{conn}} & \text{CORR} \\ R_{x-y}(t,s) \equiv \left.\dfrac{\delta}{\delta h_y(s)} \langle \phi_x(t)\rangle\right|_{h=0} & \text{RESP } \textit{(lin.)} \end{cases}$$

$$t_{\text{EQ}}(T) \ll s < t : \quad C_x(t,s) = C_x^{(eq)}(t-s) \quad ; \quad R_x(t,s) = R_x^{(eq)}(t-s)$$

$$\boxed{T R_x^{(eq)}(\tau) = -\frac{dC_x^{(eq)}(\tau)}{d\tau}} \qquad \text{FDT}$$

In general? Fluctuation-Dissipation Ratio [CK '94]

$$X(t,s) \equiv \frac{T R(t,s)}{\partial_s C(t,s)}$$

$$X^\infty \equiv \lim_{s\to\infty} \lim_{t\to\infty} X(t,s) \qquad = 1 \text{ if } t_{\text{EQ}}(T) < \infty$$

??? FDT & TD with $T_{\text{eff}} \equiv T/X^\infty$ $\overset{YES}{\rightsquigarrow}$ $\infty$-range glass

| Model (Glauber dyn) | | $X^\infty(\infty \mapsto T_c)$ |
|---|---|---|
| Random Walk, GF | [1] | $1/2$ |
| Spherical | [2] | $1-2/d$ |
| 1-dim. Ising | [2,3] | $1/2$ |
| 2-dim. Ising | [2] | 0.26(1) |
| " " | [4] | 0.340(5) |
| " " | [5] | 0.33(2) |
| " " | [6] | 0.33(1) |
| " " | [7] | 0.330(5) |
| 3-dim. Ising | [2] | 0.40 |
| 3-dim. XY | [8] | 0.43(4) |

Exact solution, Monte Carlo simulations.

**[1] Cugliandolo, Kurchan, Parisi 1994**

**[2] Godrèche, Luck 1999, 2000**

**[3] Zannetti** *et al.* **1999**

**[4] Mayer** *et al.* **2003**

**[5] Chatelain 2003**

**[6] Sastre, Dornic, Chaté 2003**

**[7] Chatelain 2004**

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Idea: $X^\infty$ at $T = T_c$ is *universal*
[Godrèche, Luck '00]

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$\Downarrow$

exploit Universality! [CG'02]

| LATTICE, $\mathbb{Z}^d$ | | CONTINUUM, $\mathbb{R}^d$ |
|---|---|---|
| Ising, $S_i(t)$<br>$O(n)$, $S_i^\alpha$ | $\overset{\textit{Scaling}}{\longmapsto}$ | LGW, $\varphi(x,t)$<br>$O(n)$ LGW, $\varphi^\alpha$ |
| 1. spin-flip | $\overset{\textit{Scaling}}{\longmapsto}$ | MODEL A |
| 2. spin-exch | $\overset{\textit{Scaling}}{\longmapsto}$ | MODEL B |

$\Downarrow$

analytical predictions for
$X(t,s)$, $X^\infty$, scaling forms, exponents, etc.

# Model A

one-component o.p. $\varphi(x,t)$ on the continuum

$$\partial_t \varphi(x,t) = -D\,\frac{\delta\mathcal{H}[\varphi]}{\delta\varphi(x,t)} + \zeta(x,t)$$

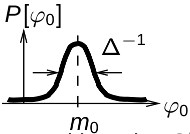
$$\langle \zeta(x,t)\zeta(x',t')\rangle = 2D\delta(x-x')\delta(t-t')$$

$$\Rightarrow P_{\mathrm{EQ}}[\varphi] \propto \mathrm{e}^{-\mathcal{H}[\varphi]}$$

$$\mathcal{H}[\varphi] = \int \mathrm{d}^d x \left[\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2} r \varphi^2 + \frac{u}{4!}\varphi^4\right]$$

**[MSR'73,BJW'76]** $\Longrightarrow S[\varphi,\tilde\varphi] = \int_0^\infty \mathrm{d}t \int \mathrm{d}^d x \left[\tilde\varphi \partial_t \varphi + D\tilde\varphi \frac{\delta\mathcal{H}}{\delta\varphi} - D\tilde\varphi^2\right]$

$$\langle \mathcal{O}\rangle_\zeta = \int [\mathrm{d}\varphi \mathrm{d}\tilde\varphi]\ \mathcal{O}\ \mathrm{e}^{-S[\varphi,\tilde\varphi]} \qquad \tilde\varphi \leftrightarrow h$$




initial cond. $\varphi_0(x,t) \equiv \varphi(x,t=0)$ with
$P[\varphi_0] \propto \mathrm{e}^{-H_0[\varphi_0]}$
$H_0[\varphi_0] = \int \mathrm{d}^d x \frac{\Delta}{2}[\varphi_0(x) - m_0]^2$

**[JSS'88]** $\Longrightarrow$ FT with *action* $S[\varphi,\tilde\varphi] + H_0[\varphi_0]$

# Scaling forms

$T = T_c, \quad s < t$

$$\begin{cases} R_{q=0}(t,s) = A_R \quad (t-s)^a \left(\frac{t}{s}\right)^{\theta} \mathcal{F}_R(s/t, t/t_0) \\ C_{q=0}(t,s) = A_C s\,(t-s)^a \left(\frac{t}{s}\right)^{\theta} \mathcal{F}_C( \quad , \quad ) \end{cases}$$

$a = (2-\eta-z)/z$
$\theta$ initial-slip exponent [JSS'88]
$\mathcal{F}_{R,C}(0,0) = 1$, universal;

$t_0 \equiv A_m m_0^{-1/\sigma}$ non-univ
$\sigma = \theta + a + \beta/(\nu z)$
$A_R$, $A_C$ non-univ.

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$\boxed{m_0 = 0}$ [JSS'88]

$$R_{q=0} = A_R (t-s)^a \left(\frac{t}{s}\right)^{\theta} \mathcal{F}_R(s/t, 0)$$

$$\overset{t \gg s}{\simeq} A_R t^a (t/s)^{\theta}$$

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$\boxed{m_0 = \infty}$ ie, $s, t \gg t_0$ **[CGK'06]**

$$R_{q=0} = a_R (t-s)^a \left(\frac{s}{t}\right)^{\beta\delta/(\nu z)} f_R(s/t) \overset{t \gg s}{\sim} a_R t^a (s/t)^{\beta\delta/(\nu z)}$$

$$X^{\infty} = \frac{a_R}{a_C(1-\frac{\beta\delta}{\nu z})}$$

$\Longrightarrow$ for long times: (a) $m_0 = 0$ (b) $m_0 \neq 0 \rightsquigarrow$ *crossover*

# Results $m_0 = 0$

Ising model, Glauber dynamics

- *Global* magnetization M(t):

| $X^\infty$ | FT |
|---|---|
| $d = 2$ | 0.30(5) |
| $d = 3$ | 0.429(6) |
| $d > 4$ | $1/2$ |

[a] Mayer,Berthier, Garrahan and Sollich (2003)....
[b] Godrèche and Luck (2000)
[c] Corberi, Lippiello and Zannetti (2006), $d = 4$

**note**: same results for *local* magnetizaton $\langle\varphi(x,t)\rangle$

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- Other observables:

$R, C, X \mapsto R_{\mathcal{O}}, C_{\mathcal{O}}, X_{\mathcal{O}}$ (eg. $\mathcal{O} = \varphi^2$ energy density, $h_{\mathcal{O}}$ bath temp.)

Gaussian fluct. $(d > 4)$: $X^\infty = X^\infty_{\mathcal{O}}$ $\forall\mathcal{O}$ local

Non-Gaussian fluct. $(d < 4)$: $X^\infty \neq X^\infty_{\mathcal{O}} \Longrightarrow$ ??? $T_{\text{eff}} \equiv T_c/X^\infty$

!! MC $X^\infty_E = 0.33(2)$([b])

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  (a) Checked scaling forms & exp. $\beta\delta/(\nu z)$, MC $d = 2$

(b)

| $X^\infty$ | FT | |
|---|---|---|
| $d = 2$ | 0.75 | ([a]) |
| $d > 4$ | $4/5$ | |

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- $O(n)$, $n > 1$: *unphysical dyn.*
  Gaussian fluc.: $X_L^\infty = 4/5$, $X_T^\infty = 2/3$ $\Longrightarrow$ no $T_{\rm eff}$ even at the Gaussian level

- Purely dissipative dyn (Model A), $O(n)$ GLW:
  - $n = 1$ *anisotropic magnets/alloys*
  - $\forall n$ lattice spin models with Glauber dyn
  - $\rightsquigarrow$ $X^{\infty}(m_0 = 0)$[2L]: OK w. MC Ising 2d & 3d and XY 3d
  - $\rightsquigarrow$ $X^{\infty}(m_0 \neq 0)$[1L]: OK w. MC Ising 2d
  - $\rightsquigarrow$ $T_{\mathrm{eff}}$ ???
- Conserved dyn (Model B) scalar GLW
  - some *uniaxial ferromagnets*
  - Ising model with Kawasaki dyn
  - $\rightsquigarrow$ [1L] $X^{\infty}_{\mathrm{Model\ B}} > X^{\infty}_{\mathrm{Model\ A}}$: OK with MC Ising 2d **[GKRT'04]**
- Other dyn: Model C [1L], Model A dilute Ising [1L] **[SP'04]**

# . . . what should be looked at. . .

- ? Aging at surfaces (different Universality Classes!) **[MC:P'05; Sph mod:PB'05]**
- ? More realistic dynamics...
  - magnets $\Longrightarrow$ Models J,G (A)
  - fluids $\Longrightarrow$ Model H (B)

  . . . **waiting for experiments!**
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**Take-home message:**

FT is a viable approach to investigate aging phenomena at critical points!

Ref.: Calabrese&G, J.Phys.A **38** (2005) R133-R193