

# **Dynamical order in chaotic Hamiltonian system with many degrees of freedom**

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  - **Kunihiko Kaneko (Tokyo Univ.)**

## Abstract

In this talk we introduce several systems which have ordered structure such as clusters by its own dynamics. The systems consists of particles with long-range interaction, just like many-body systems in astrophysics. Since the systems are Hamiltonian systems, the spatial structures thus formed are not “attractors” or asymptotic states we observe in the infinite future. Rather the states are observed in transiency or in the course of itinerancy among several quasi-stationary states.

# 1 Part I : Introduction

**(my) basic, very primitive question:**

**“What the world is made of?”**

**[picture: Andromeda galaxy]**

**[picture: hemoglobin molecule]**

- **Various structures exist in every scale.**
- **The world is not uniform.**

## 2 statistical description

**Way to describe structure**

**most successful : (equilibrium) statistical mechanics (+ phase transition theory)**

### 3 basis of statistical mechanics

**(total system) = (system to be observed) + (a large system called “heat bath”)** (1)

- **total system is closed, conserved**
- **requiring principle of equal weight for total system  $\Rightarrow$  canonical prob. distribution for the system to be observed**
- **(note: equal weight in the total system  $\Rightarrow$  “equipartition of energy”  $\langle \frac{p^2}{2m} \rangle = \frac{k_B T}{2}$  for the system to be observed .)**

### 3.1 ergodicity

def. **time average = phase space average**

$$\bar{A} = \langle A \rangle \quad (2)$$

equivalent def. **invariant set is the whole phase space itself (or empty set).**

**(regardless of initial condition one can visit any state.)**

**[Arnold and Avez]**

**sometimes, ergodicity fails to be satisfied...**

## 4 demonstration 1: standard map

### standard map

$$(x, p) \mapsto (x', p')$$

$$p' = p + \frac{K}{2\pi} \sin(2\pi x)$$

$$x' = x + p' \pmod{1} \quad (K > 0)$$

**visiting phase point in very much non-uniform way ( $1/f$ -type fluctuation, sticking/stagnant motion)**



5 discrete standard map : investigating non-uniformity of phase space

**Rannou(1984), TK(2006)**  
**standard map**

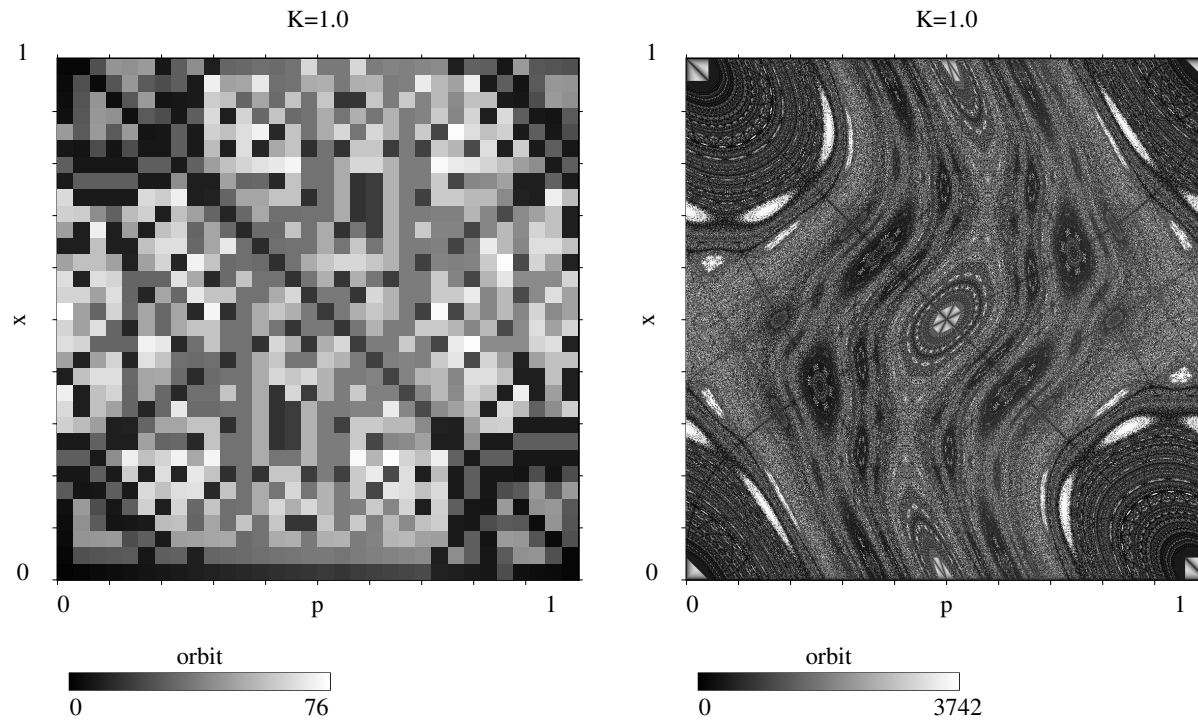
$$(I_x, I_p) \mapsto (I'_x, I'_p), \quad I_x \text{ etc.} \in 0, 1, 2, \dots, M$$

$$I'_p = \left[ I_p + M \cdot \frac{K}{2\pi} \sin\left(\frac{2\pi I_x}{M}\right) \right]$$

$$I'_x = I_x + I'_p \pmod{M} \quad (K > 0)$$

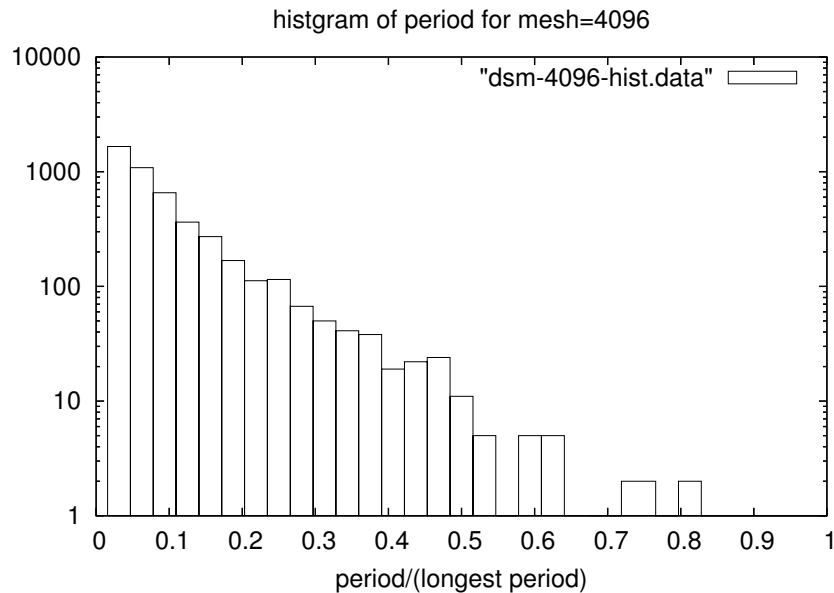
( $[\dots]$ ): nearest integer)

- 1 to 1
- every orbit is periodic

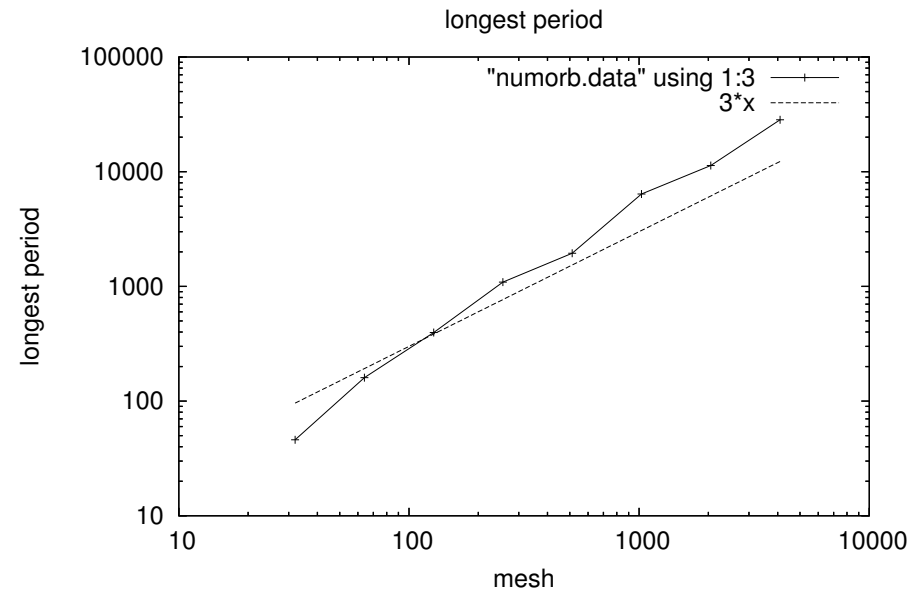


**discrete standard map : left:low resolution ( $M = 32$ ), right: increased resolution ( $M = 1024$ )**

# statistics of orbits:



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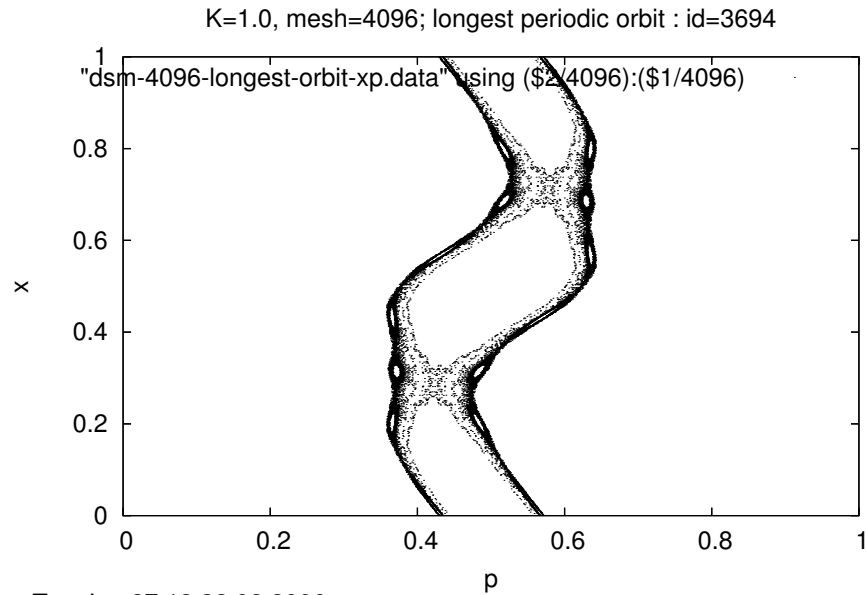


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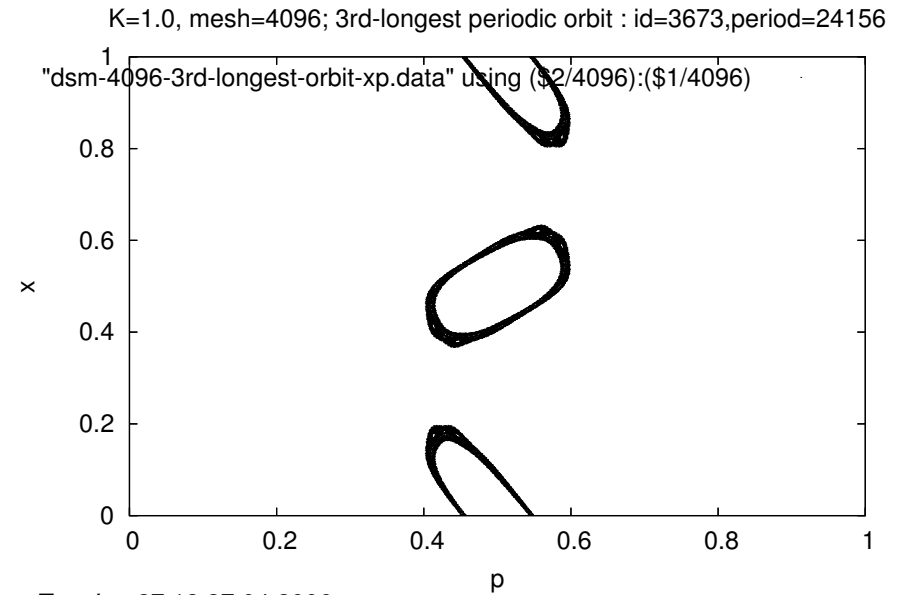
**(left) distribution of periods of orbits (resolution=4096)**  
**(right) longest period vs. resolution**

- no dominant orbit appears

# dynamics:



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**(left) longest periodic orbit ( $M = 4096$ )**

**(right) 3rd longest periodic orbit ( $M = 4096$ )**

6 many body system

**failure of ergodicity in small system: pathology caused by limitation of deg. of freedom?**

**many body system helps?**

## 7 failure of ergodicity in many body systems

### 7.1 trivial example

- **free particle system**

$$H = \sum_{i=1}^N \frac{p_i^2}{2m_i} \quad (3)$$

- **harmonic chain**

$$H = \sum_{i=1}^N \frac{p_i^2}{2m} + \sum_{i=1}^N \frac{k}{2} (x_{i+1} - x_i)^2 \quad (4)$$

$\Rightarrow$  every mode energy is conserved :  $E_k(t) = E_k(0)$

## 7.2 less trivial example

- **1-dim. linear chain with quartic or quadratic interaction (AKA Fermi – Pasta – Ulam model)**

$$H = \sum_{i=1}^N \frac{p_i^2}{2m} + \sum_{i=1}^N \frac{k}{2} (x_{i+1} - x_i)^2 + \beta \sum_{i=1}^N \frac{1}{4} (x_{i+1} - x_i)^4 \quad (5)$$

**However, numerical simulation tells us that the mode energies are recurrent for the system (no relaxation).**

$$E_1(t = 0) = E_{total}, E_k(t = 0) = 0 (k \geq 2)$$

$$\Rightarrow \begin{cases} \text{(wrong)} & E_1 = E_2 = \dots = E_N \\ \text{(true)} & E_1(T) \sim E_0, E_k \ll E_0 (k \geq 2) \end{cases}$$

**Later analysis revealed that the model (5) is quite close to an *integrable equation* of 1-dim field “modified KdV equation”.**



## 8 Statistical mechanics: successful application to explain structure

**Anyway, removing some difficulties, statistical mechanics succeeded in explaining order and structures:**

- **ferromagnetic transition for Ising model and many other spin systems**
- **helix-coil transition for long chain of molecules**

**huge number of examples..**

Statistical mechanics assumes:

- the system is in thermal equilibrium
- there may be some perturbations applied to the system , but they relaxes quickly: in other words, there exists a characteristic relaxation time  $\tau_{relax}$  and the perturbations relaxes in time as

$$\propto \exp\left(-\frac{t}{\tau_{relax}}\right)$$

**downside : fails to describe structures which are not stationary**

## Examples:

- dynamics of water molecules (Ohmine group, Sasai group)
- Elliptical galaxies

`http://www.chem.nagoya-u.ac.jp/~og/`

`10Research/10Dynamics/Gallery/index.phtml.en`

**M. Sasai et al., J. Chem. Phys. 96(1992) 3045.**

Purpose of this talk:

- **As we have seen, spatial structures are not necessarily in equilibrium**
- **Then what can we do?**
- **The structures are created from their own dynamics, so we hope we can understand something from dynamics**

**By presenting some examples where spatial structure are created from dynamics, we will search for methods and concepts to understand the dynamical origin of spatial structures.**

# Part II : Clustered motion in globally coupled symplectic map

Refs. TK and K. Kaneko, J. Phys. A 25 (1992) 6283

K. Kaneko and TK, Physica D 71 (1994) 146

- model
- clustered motion and random motion
- Lyapunov spectra – “minimum” exponent?
- Fluctuation property of Lyapunov exponent
- Phase space structure – order supported by chaos

## Model

$$\left(x_i, p_i\right) \mapsto \left(x'_i, p'_i\right), \quad i = 1, 2, \dots, N,$$

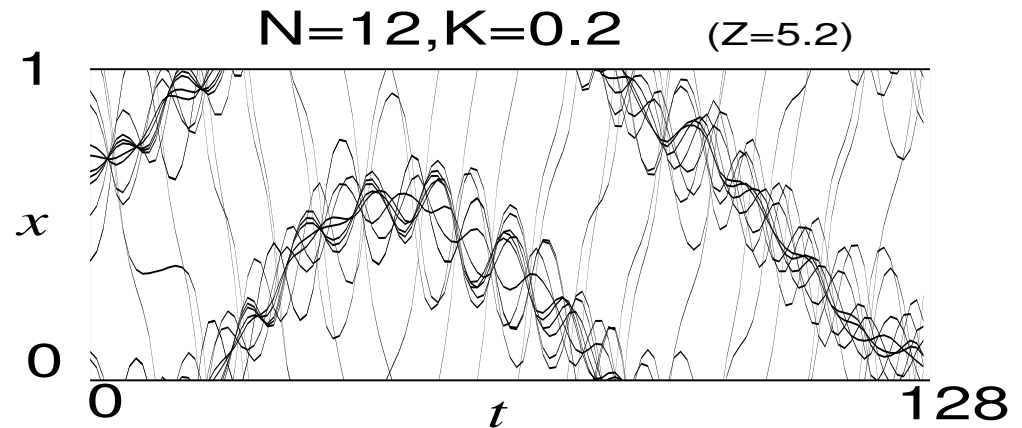
$$p'_i = p_i + K \sum_{j=1}^N \sin 2\pi(x_j - x_i), \quad (6)$$

$K > 0$  : attractive,  $K < 0$  : repulsive

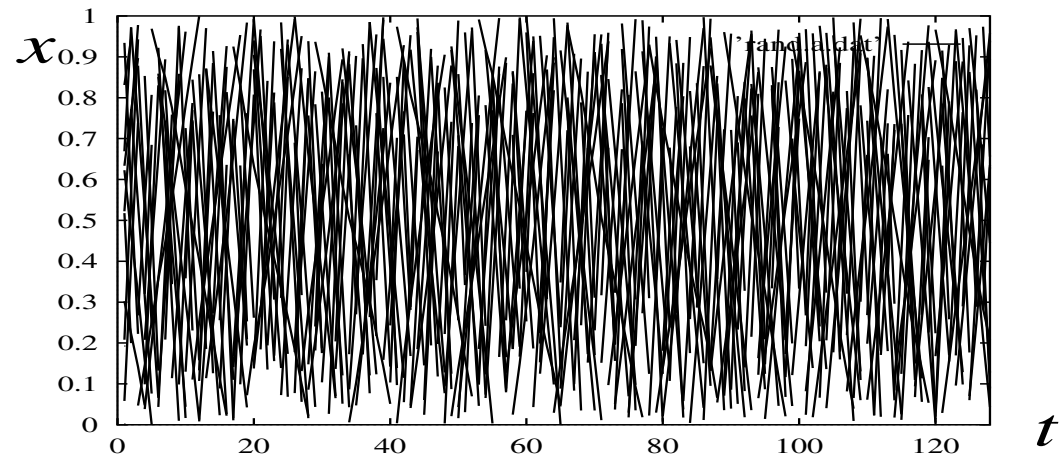
$$x'_i = x_i + p'_i.$$

two phases of motion (with same  $K$ )

clustered :



uniformly random:

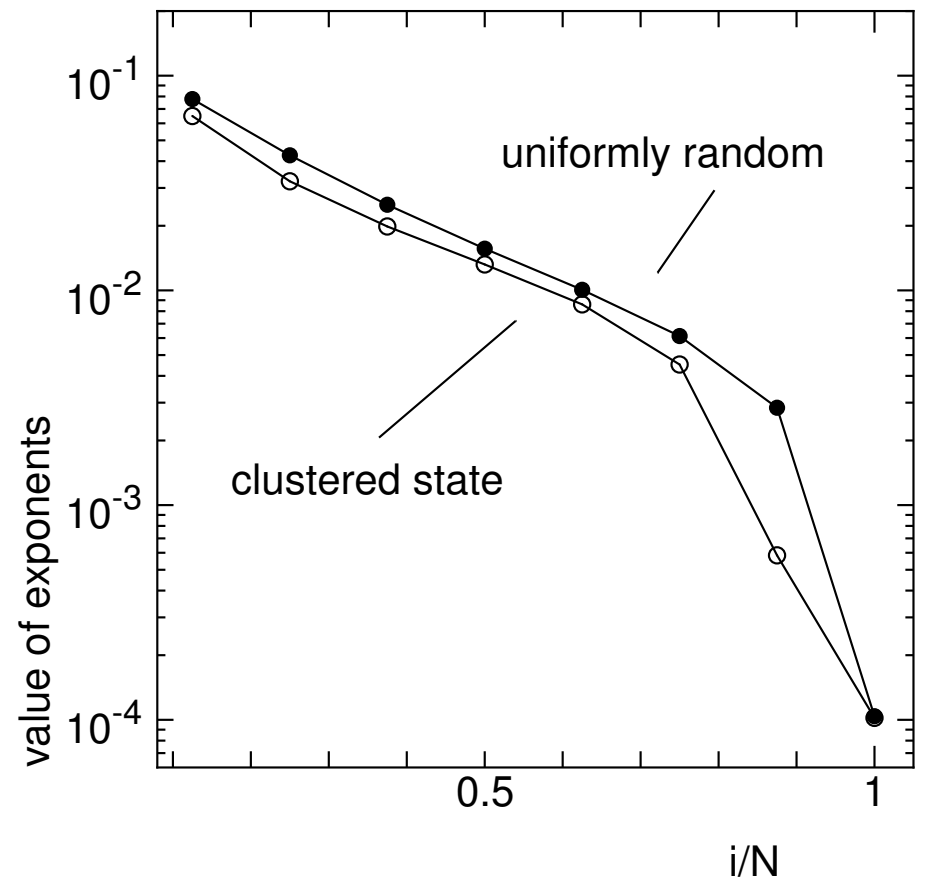


$$p'_i = p_i + K \sum_{j=1}^N \sin 2\pi(x_j - x_i), \quad K : \text{real}$$

# Lyapunov spectra

- Both phases are chaos.
- difference of Lyap. exp. is observed only at the  $\lambda_{N-1}$ .  
 $\Rightarrow$  This subspace represents instability of cluster?

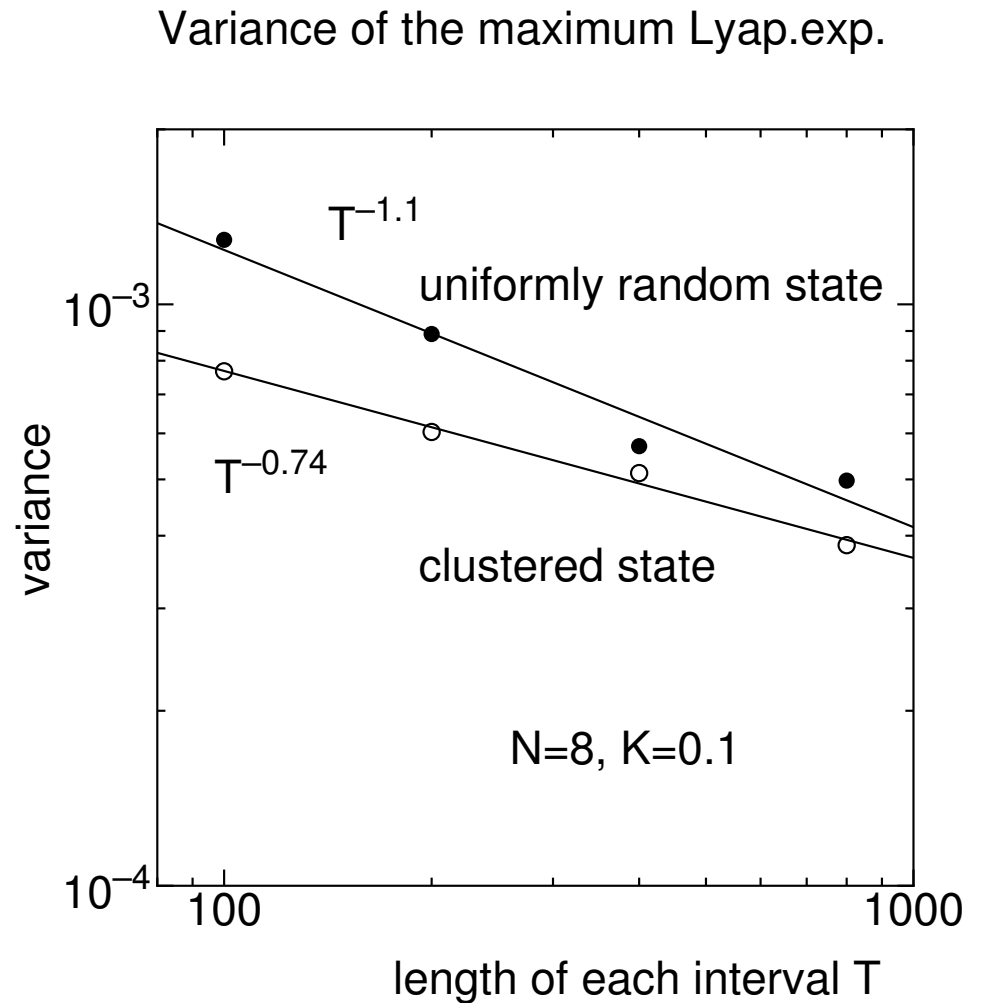
Lyapunov spectrum : N=8, K=0.1





# Fluctuation property of Lyapunov exponent

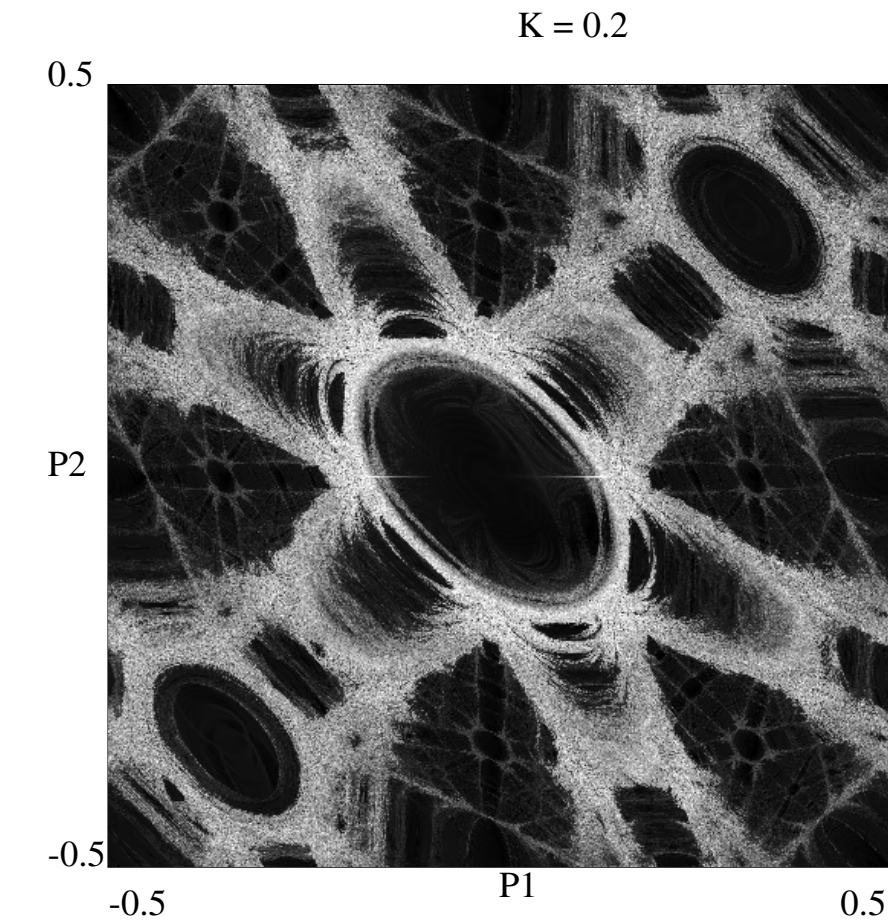
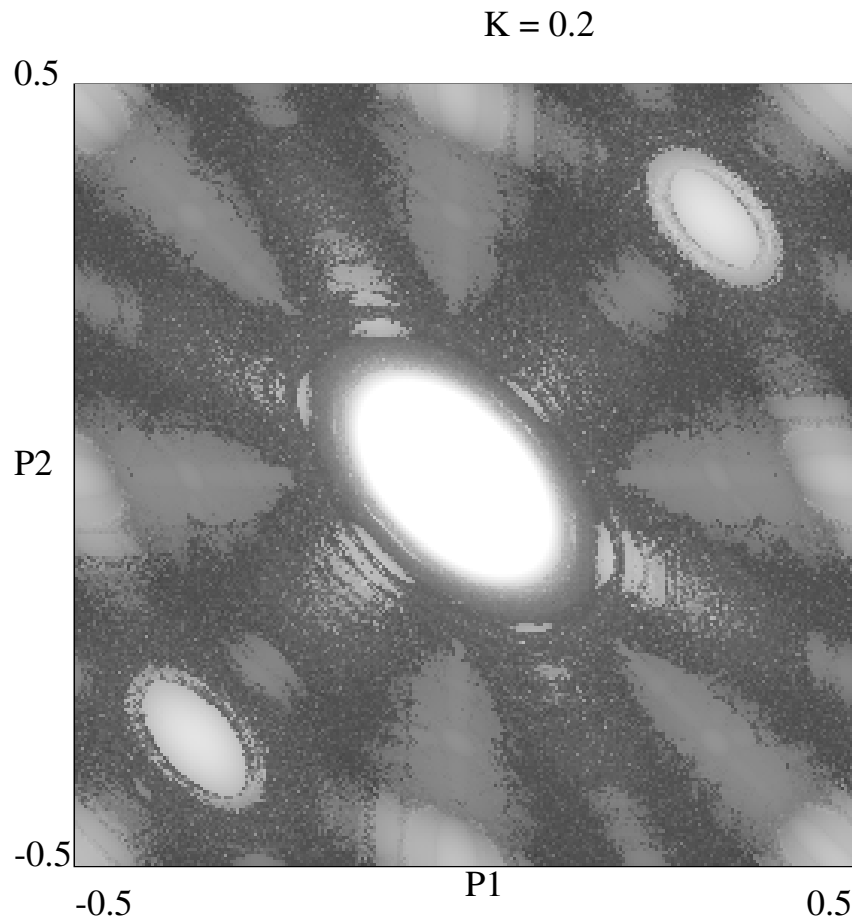
- clustered state ... strong temporal correlation
- non-clustered state ... no temporal correlation



phase space structure

init. cond. for clustered motion (white)

local  $\lambda_1$  ( black  $\dots$  stable)



- Self-similar structures in the phase space – common to most Hamiltonian systems
- chaos supports the order

# Part III : Emergence of power-law correlation in the mass-sheet model

Refs. H. Koyama and TK, Phys. Lett. A vol. 295 (2002) 109

H. Koyama and TK, Europhys. Lett., vol. 58(2002) 356.

H. Koyama and TK, Phys. Lett. A vol.279 (2001) 226.

model

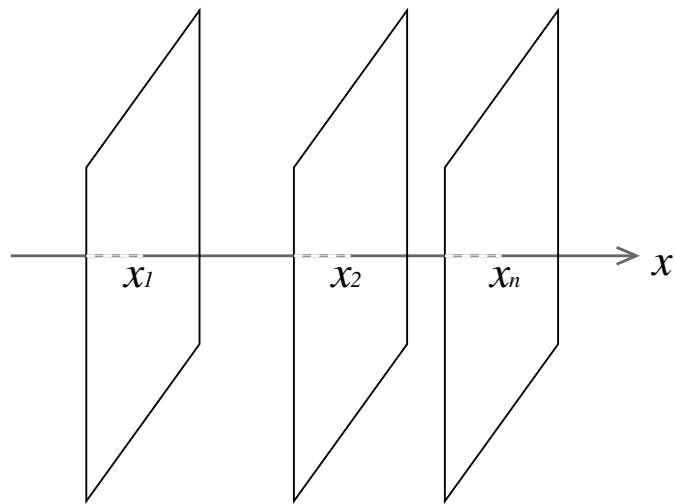


Fig. 1 A schematic picture of the model (7)

$$H = \sum_{i=1}^N \frac{p_i^2}{2m} + 2\pi Gm^2 \sum_{i>j} |x_i - x_j| , \quad -\infty < x_i < \infty , \quad (7)$$

# formation of fractal structure

dynamics seen in  $\mu$ -space

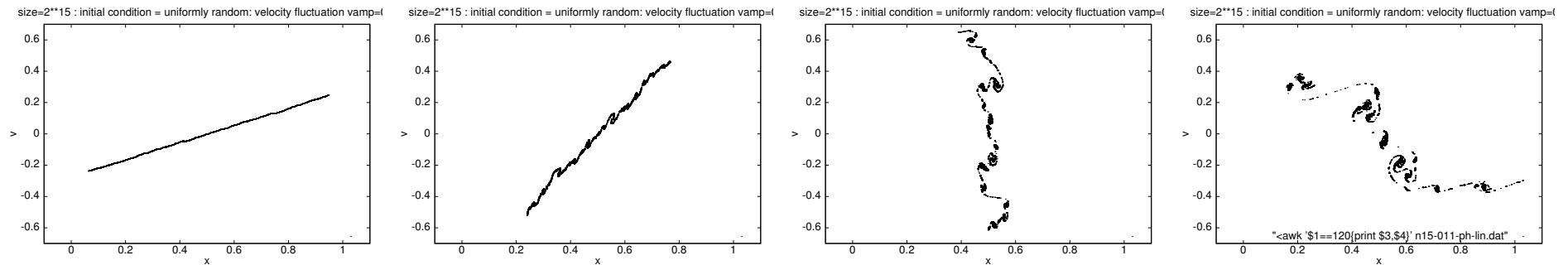


Fig. 2 Formation of fractal structure. Initial condition :  $x_i$  : uniformly random in  $[0,1]$ ,  $u_i = 0$ ,  $N = 2^{15}$ . Time are 2.34375, 4.6875, 7.03125, and 9.375.

# box-counting dimension

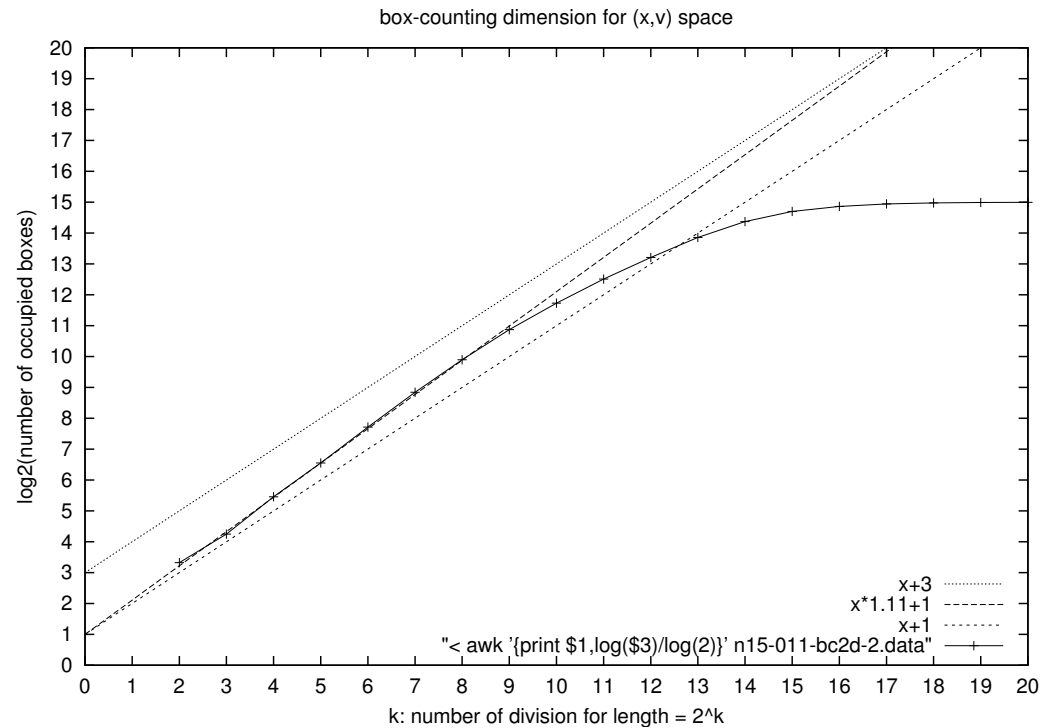


Fig. 3 Box counting dimension of the  $\mu$ -space distribution shown in the bottom of Fig.2. Sample orbits with the same class of initial condition with different random number gives dimension  $D = 1.1 \pm 0.04$  for 24 samples. Lines with  $D = 1$  are also shown for comparison.

## 2-body correlation function $\xi(r)$

$$dP = ndV(1 + \xi(r)) \quad (8)$$

$$\xi(r) \propto r^{-\alpha} \quad (9)$$

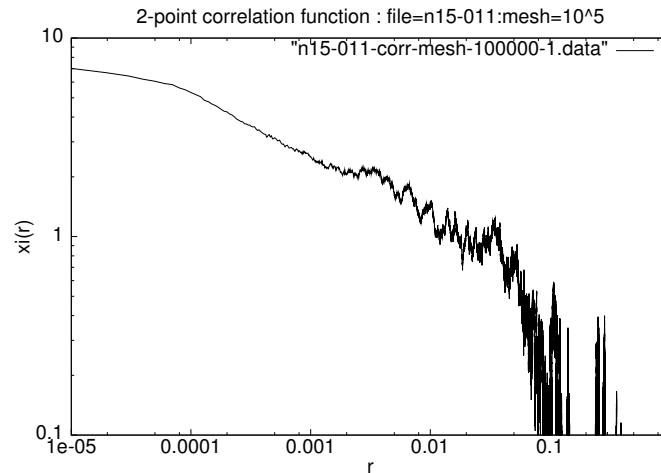


Fig. 4 two-point correlation function  $\xi(r)$  for  $t = 9.375$  in Fig. 2. Exponent  $\alpha$  of  $\xi \propto r^{-\alpha}$  is  $\alpha = 0.20 \pm 0.03$  for 24 samples.



# development of power-law structure

spatial scale : small  $\rightarrow$  large ( “hierarchical clustering” )

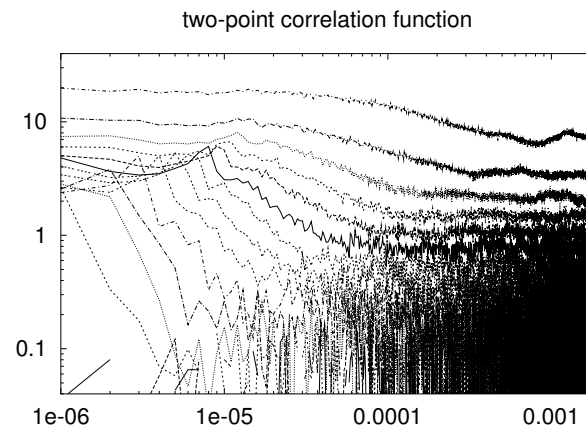


Fig. 5 Evolution of correlation function  $\xi(r)$  at  $t = \frac{5}{64}\ell$ ,  $\ell = 3, 4, \dots, 17$  (from bottom to top). System size  $N = 2^{14}$ , Initial condition:  $x_i = \text{random}$ ,  $v_i = 0$ .

## relaxation of the structure

Actually, the power-law structure  $\xi(r) \propto r^{-\alpha}$  decays, but long after virialization.

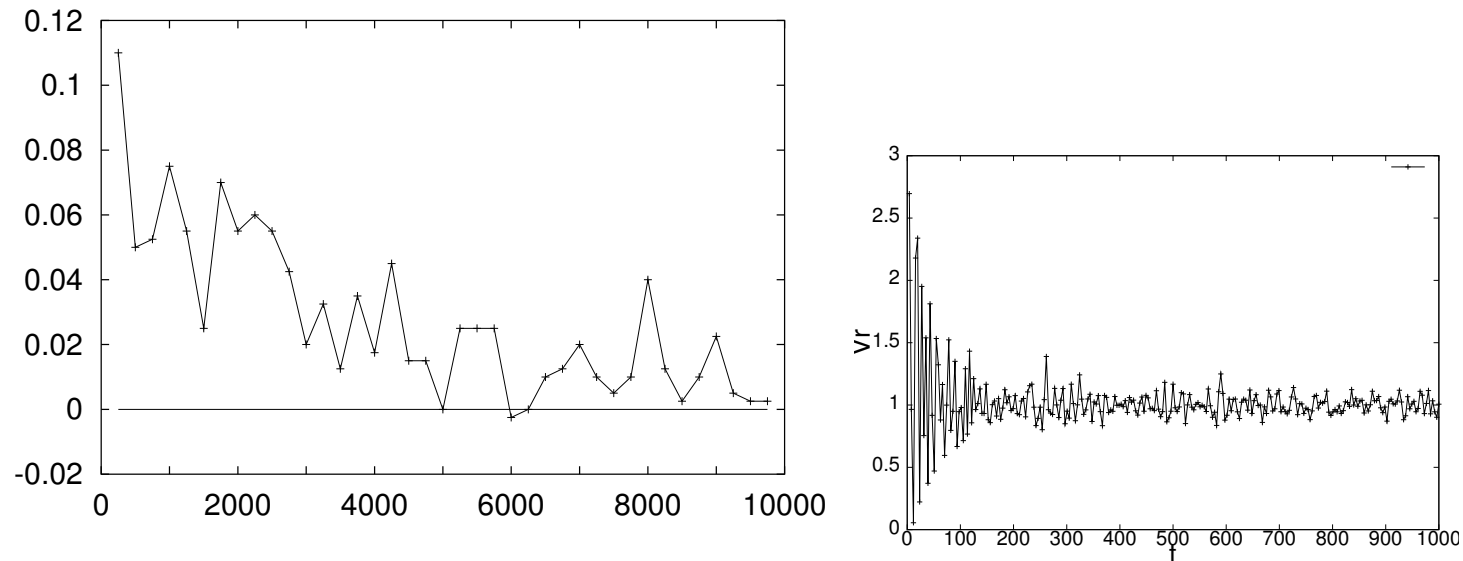


Fig. 6 time dependence of the exponent  $\alpha(t)$  (left) and the virial ratio  $2E_{kin}/E_{pot}$  (right) (9)

## Part IV : equilibrium and nonequilibrium behavior of excitation in HMF model

- Koyama, Ruffo, TK (2006)
- Koyama, talk at this conference

$$H = \sum_{i=1}^N \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^N (1 - \cos(\theta_i - \theta_j)) \quad (10)$$

- cluster formation
- excitation of high-energy particle

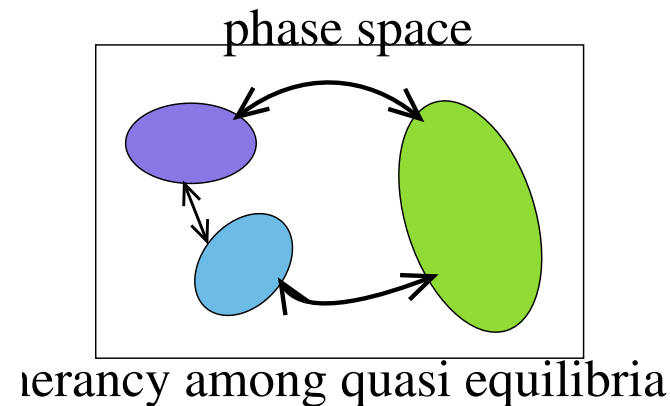
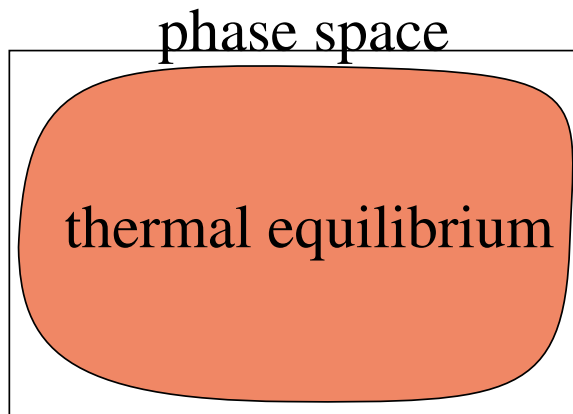
trapping ratio :

$$R \equiv \frac{N_{LowEnergyParticle}}{N_{total}} \quad (11)$$

- trapping ratio is well described by equilibrium statistical mechanics
- on the other hand, lifetime of fully-clustered state (state without any high energy particles) obeys power law

# Summary

Spatial structures can emerge in non-equilibrium situations. Their origins are non-uniformity of the phase space, where dynamical properties of the system is reflected:



equilibrium structure (left) and dynamical order(right)

- chaos does not necessarily imply “mess” nor “darkness”: they can be rich sources of dynamic order.

- long-range interaction may be important in creating structure/order which are dynamically changing.