2-color QCD with finite chemical potential: the method of analytical continuation

P. Cea[1,2], L. Cosmai[2], M. D'Elia[3] and A. Papa[4]

[1]Dipartimento di Fisica, Università di Bari

[2]INFN-Bari

[3]Dipartimento di Fisica, Università di Genova and INFN-Genova

[4]Università della Calabria & INFN - Cosenza

SM & FT 2006 - Bari, September 20-22, 2006

Outline

Outline

Outline

Outline

Introduction and motivation

- Understanding the phase diagram of QCD on the temperature – chemical potential (T, μ) has many important implications in cosmology, in astrophysics and in the phenomenology of heavy ion collisions.

- The discretization of QCD on a space-time lattice and the use of Monte Carlo numerical simulations in the Euclidean space-time provide us with a useful investigation tool.

- However, in QCD with non-zero chemical potential, however, the fermion determinant becomes complex and standard numerical simulations are not feasible – the so-called sign problem.

Introduction and motivation

- Ways out:
  - to perform simulations at μ=0 and to take advantage of physical fluctuations in the thermal ensemble for extracting information at (small) non-zero μ, after suitable reweighting;

[I.M. Barbour et al., 1998]
[Z. Fodor and S.D. Katz, 2002 $\rightarrow$]

Introduction and motivation

- Ways out (cont'd)
  - to Taylor expand in μ the v.e.v. of interest and to calculate by means of numerical simulations at $\mu = 0$ the coefficients of the expansion;

[S.A. Gottlieb, 1987]
[Ph. de Forcrand et al., 1999]
[A. Hart, M. Laine, O. Philipsen, 2000-2001]
[R.V. Gavai and S. Gupta, 2003-2005]
[C.R. Allton et al., 2002-2003-2005]

Introduction and motivation

- Ways out (cont'd)
  - to perform numerical simulations at *imaginary* chemical potential, for which the fermion determinant is again real, and to analytically continue the results to real μ

[M.G. Alford, A. Kapustin, F. Wilczek, 1999]
[M.P. Lombardo, 2000]

advantage: not limited by increasing lattice sizes
drawback: the extent of the attainable domain with real μ is limited
(1) by the periodicity and the non-analyticities present for imaginary μ
(2) by the accuracy of the interpolation of data for imaginary μ;

Introduction and motivation

For all the listed methods, the applicability region is restricted to $\mu/T \lesssim 1$.

- Here, we test the method of analytical continuation in a theory which does not suffer the sign problem, 2-color QCD, and show that the method can be improved considerably by using a suitable interpolation of the data at imaginary μ.

Outline

QCD with finite chemical potential

- On the continuum:

$$\mathcal{L} = \mathcal{L}_{QCD} + \mu J_0 \ , \qquad J_\mu = \overline{\psi}\gamma_\mu\psi$$

$$\int d^3x J_0 = N - \overline{N} \ , \qquad N\,(\overline{N}) \text{ no. of (anti-)particles}$$

- On the lattice:

$$U_4(n) \to e^{a\mu} U_4(n) \ , \qquad U_4^\dagger(n) \to e^{-a\mu} U_4^\dagger(n)$$

[F. Karsch, P. Hasenfratz, 1983]

$$\langle O \rangle = \frac{\int DU\, D\overline{\psi} D\psi\, O[U,\psi,\overline{\psi}]\, e^{-S_F[U,\psi,\overline{\psi}]-S_G[U]}}{\int DU\, D\overline{\psi} D\psi\, e^{-S_F[U,\psi,\overline{\psi}]-S_G[U]}}$$

$$S_F = \sum_{n,m} \overline{\psi}(n) M_{nm} \psi(m) \quad \longrightarrow \quad \int D\overline{\psi} D\psi\, e^{-S_F[U,\psi,\overline{\psi}]} = \det M[U]$$

Outline

The "sign" problem

$$\langle O \rangle = \frac{\int DU \, \langle O \rangle_{S_F} e^{-S_{eff}[U]}}{\int DU \, e^{-S_{eff}[U]}} \qquad \langle O \rangle_{S_F} = \frac{\int D\overline{\psi} D\psi \, O[U,\psi,\overline{\psi}] \, e^{-S_F[U,\psi,\overline{\psi}]}}{\int D\overline{\psi} D\psi \, e^{-S_F[U,\psi,\overline{\psi}]}}$$

$$S_{eff}[U] = S_G[U] - \ln \det M[U]$$

- In order to perform numerical simulations "det M" must be real
  - OK for $\mu = 0$ in SU(3)
  - NO for $\mu \neq 0$ in SU(3) !
  - OK for finite isospin density
  - OK for $\mu \neq 0$ in SU(2), owing to $M^* = \tau_2 M \tau_2$
  - OK for $\mu = i\mu_I$ in SU(N_c) (imaginary chemical potential)

Outline

QCD with imaginary chemical potential

- SU(N_c) gauge theory with imaginary μ

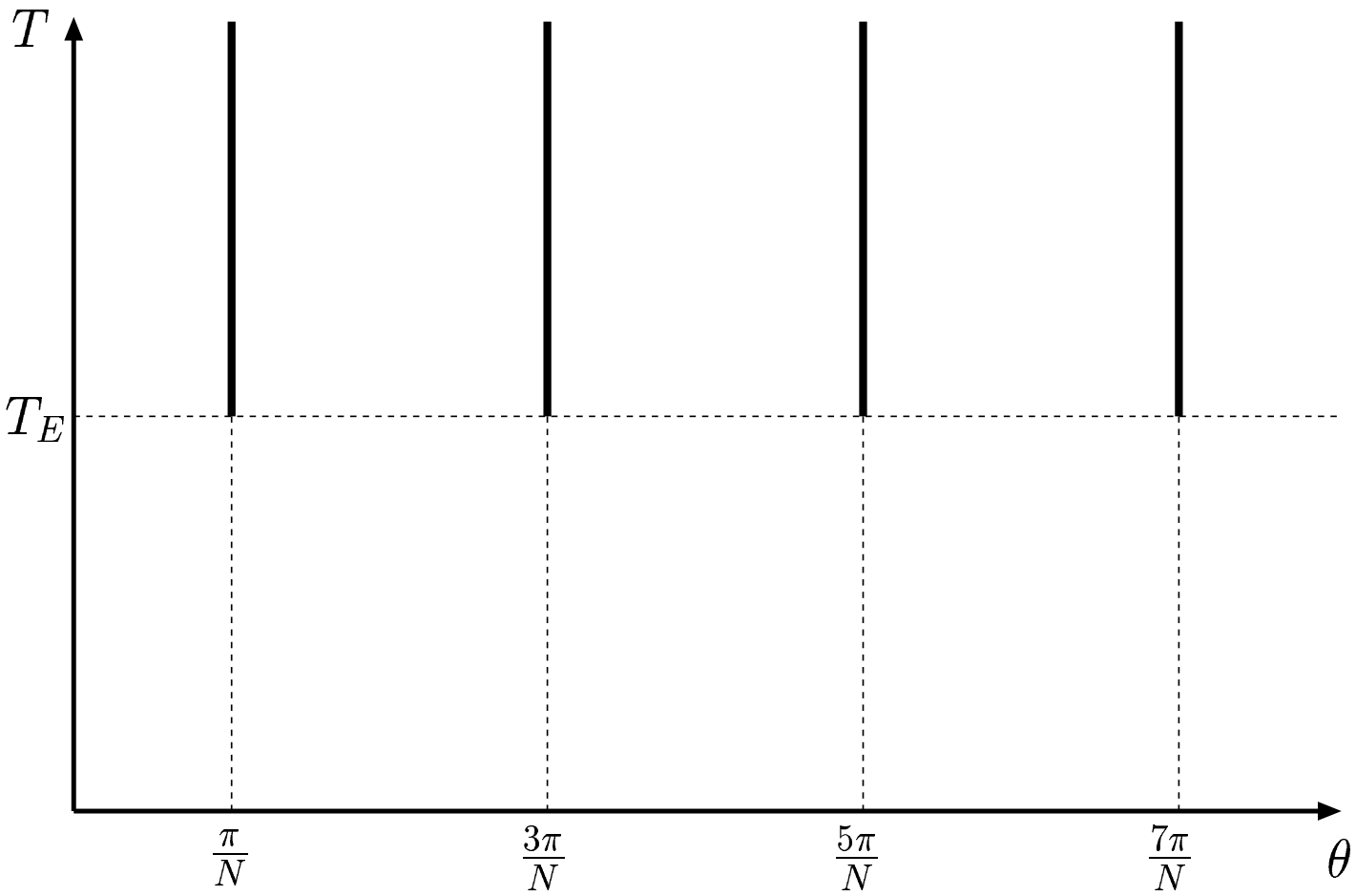
$$\mu \to i\nu, \qquad Z(\theta) = \mathrm{Tr}\left[e^{-\beta H + i\theta \hat{N}}\right], \qquad \theta = \beta\nu$$

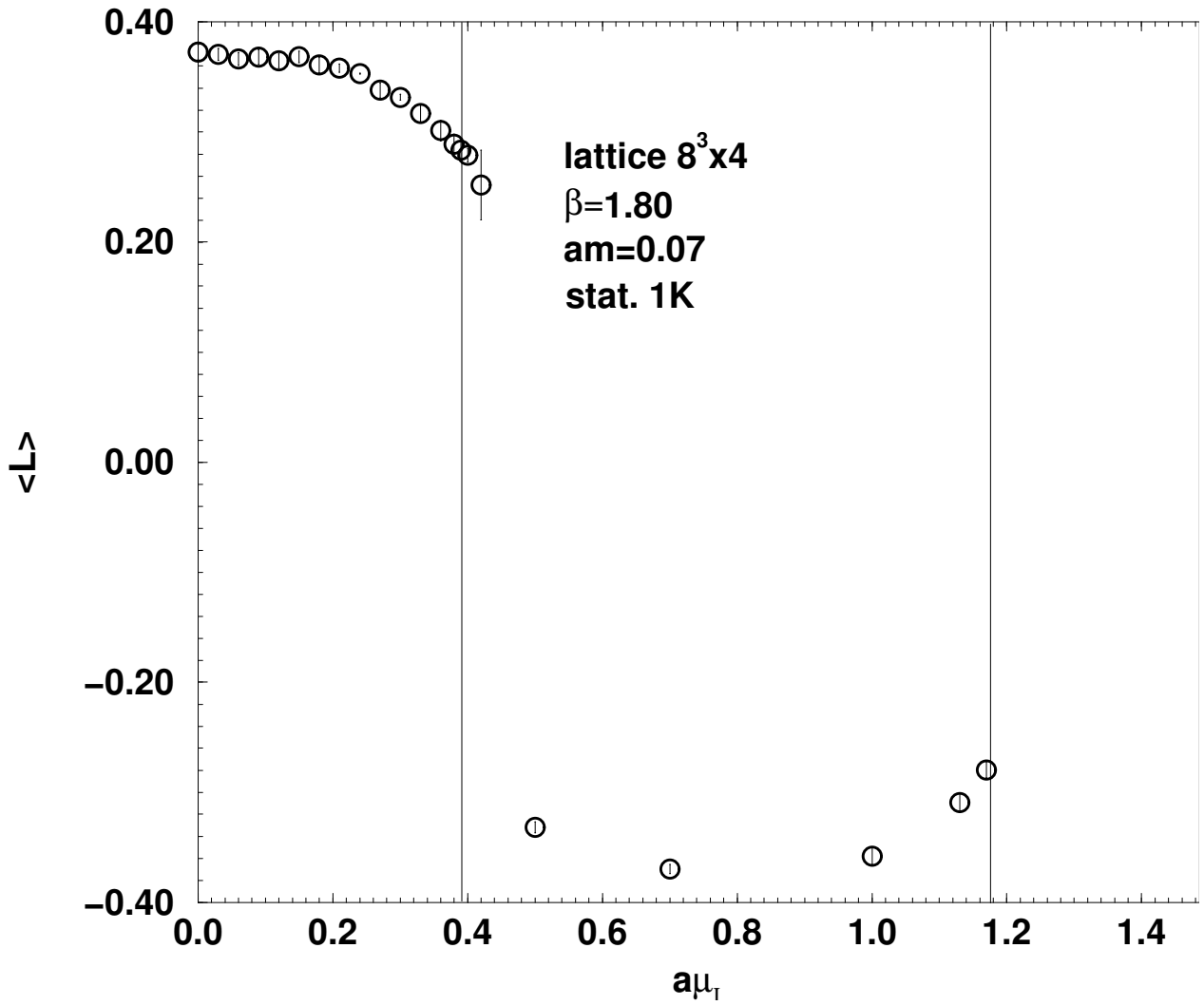
  - Free quarks ($N = 0, 1, 2, \ldots$) $\longrightarrow$ $Z(\theta)$ periodic with 2π
  - Color singlets (N multiple of N_c) $\to$ $Z(\theta)$ periodic with $2\pi/N_c$

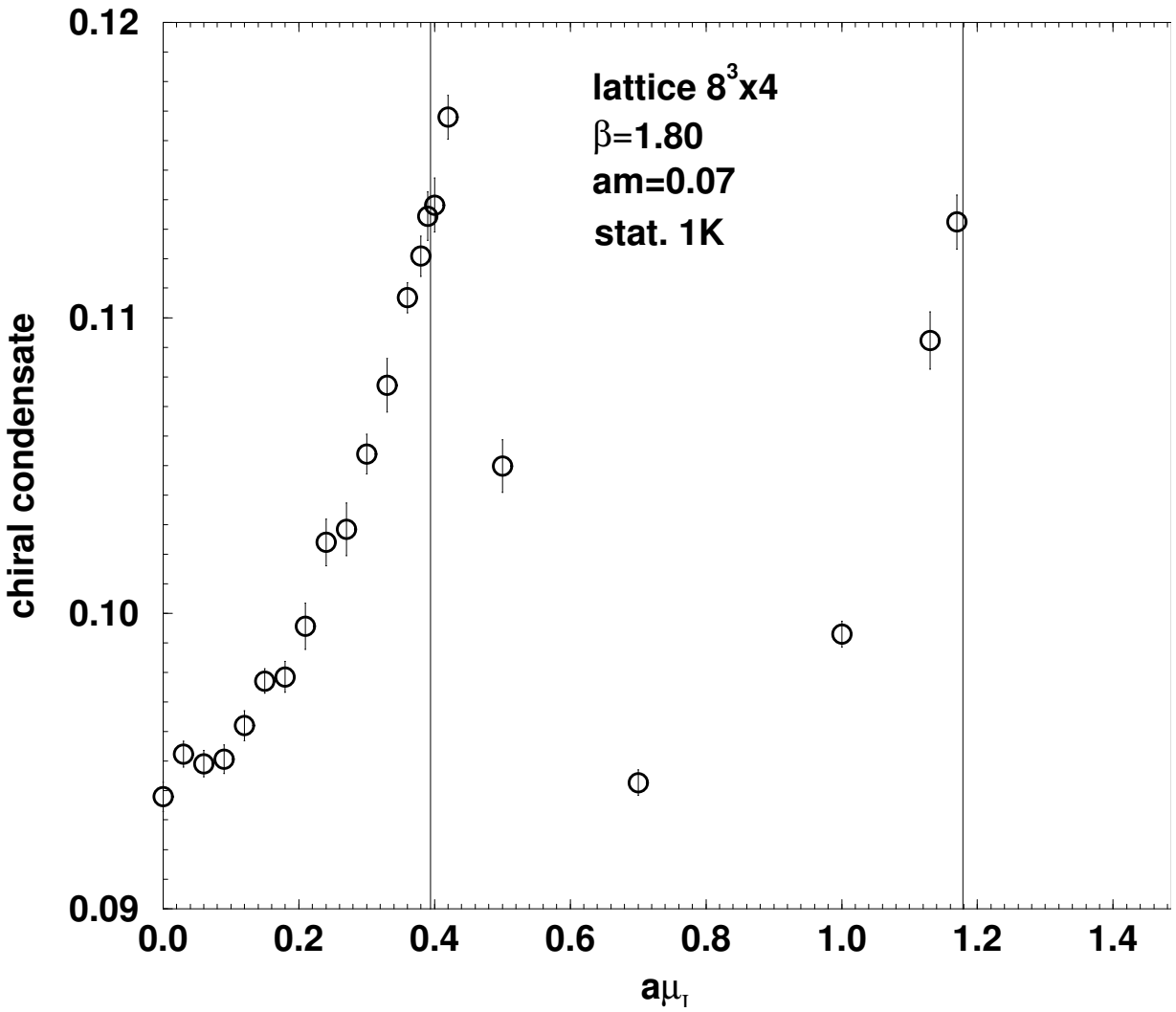
- [Roberge and Weiss, 1986] have shown that
  - $Z(\theta)$ is always periodic with $2\pi/N_c$
  - the free energy, $F(\theta) = -\ln Z(\theta)/\beta$,
    is a regular function of θ for $T < T_E$
    is a discontinuous function in $\theta = 2\pi(k + 1/2)/N_c$ for $T > T_E$

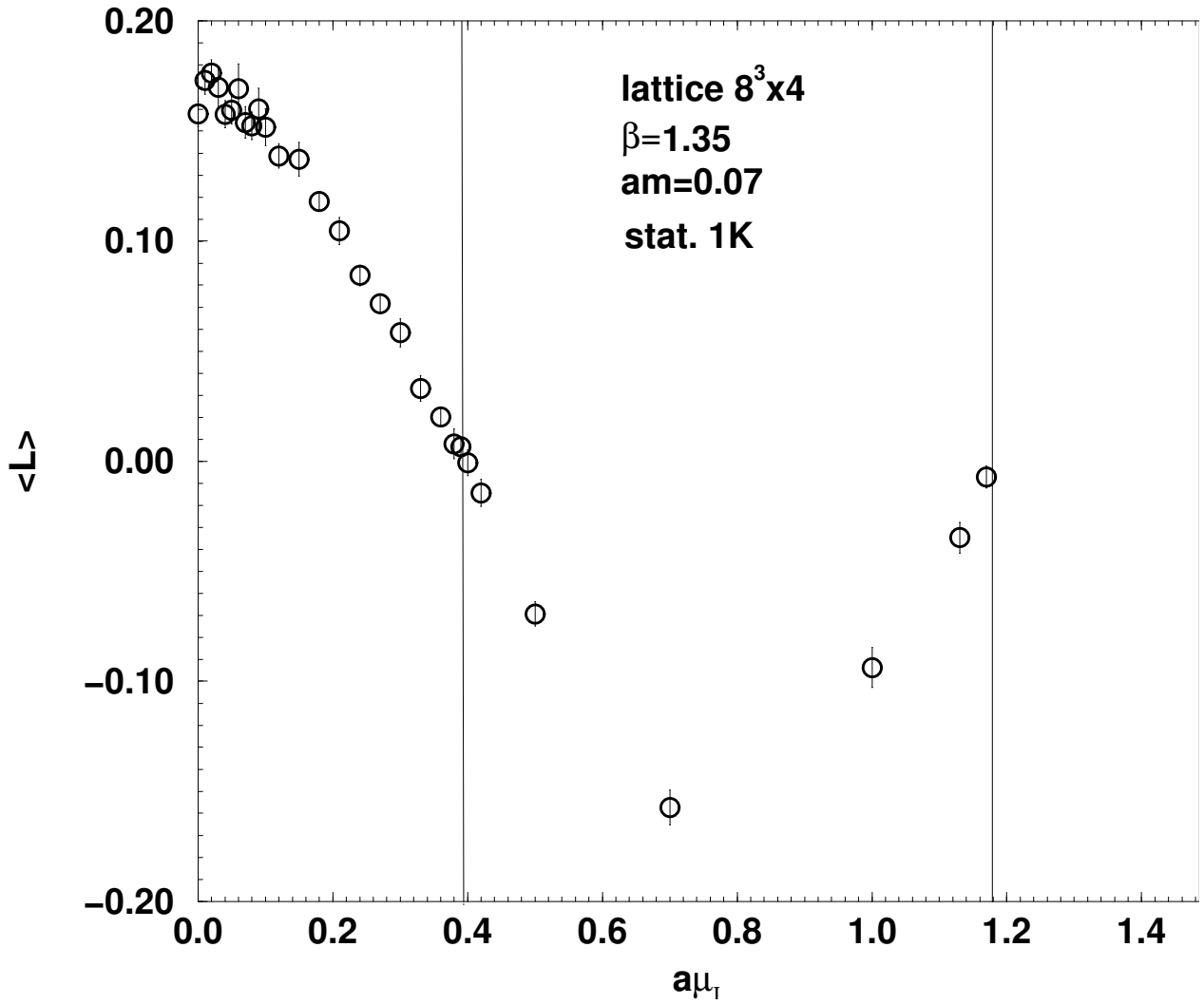
- This scenario has been confirmed in numerical simulations in SU(3) [Ph. de Forcrand and O. Philipsen, 2002; M. D'Elia, M.P. Lombardo, 2003] and in SU(2) [P. Giudice, A.P., 2004]

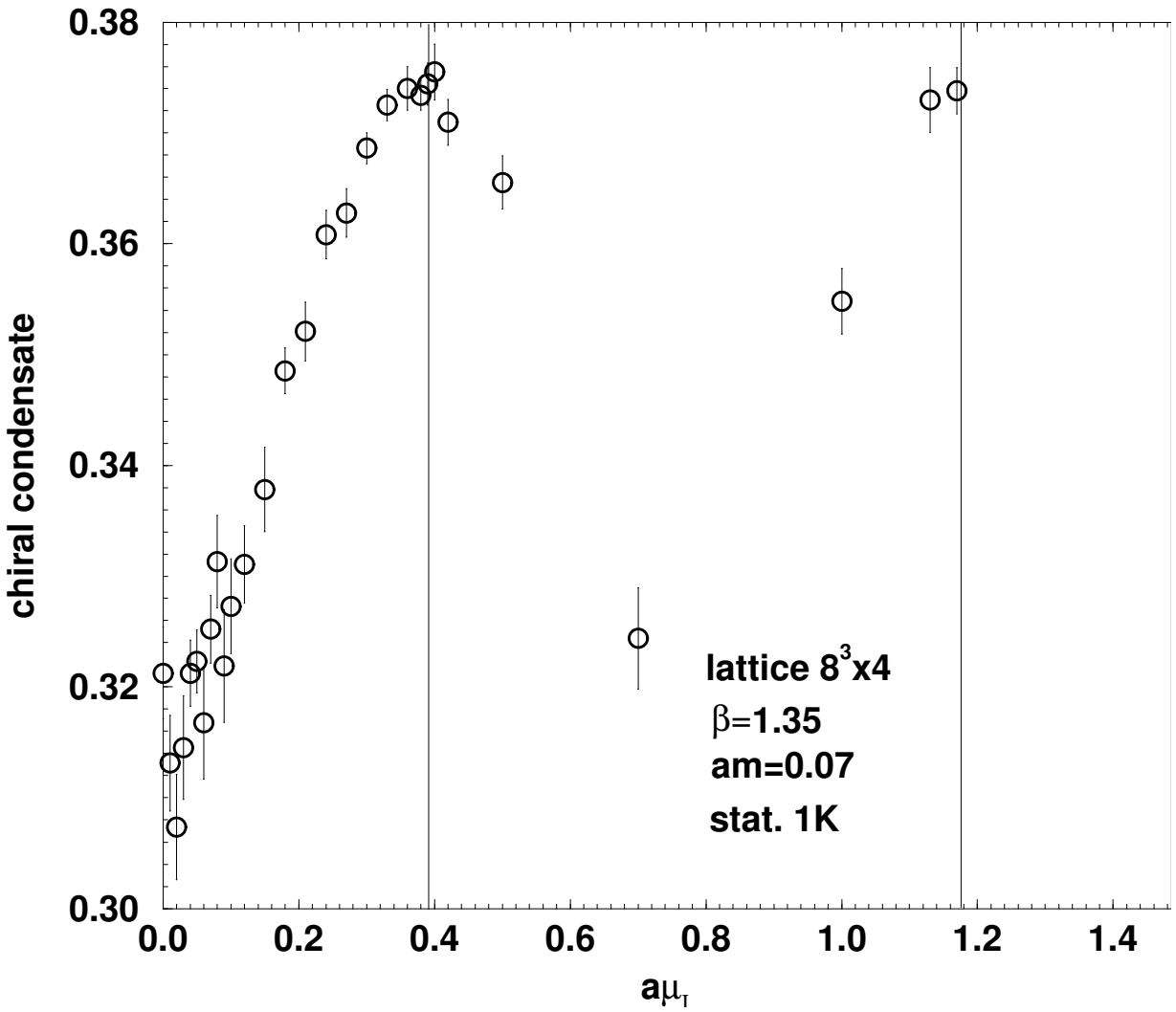
Phase diagram on the (T,θ)-plane



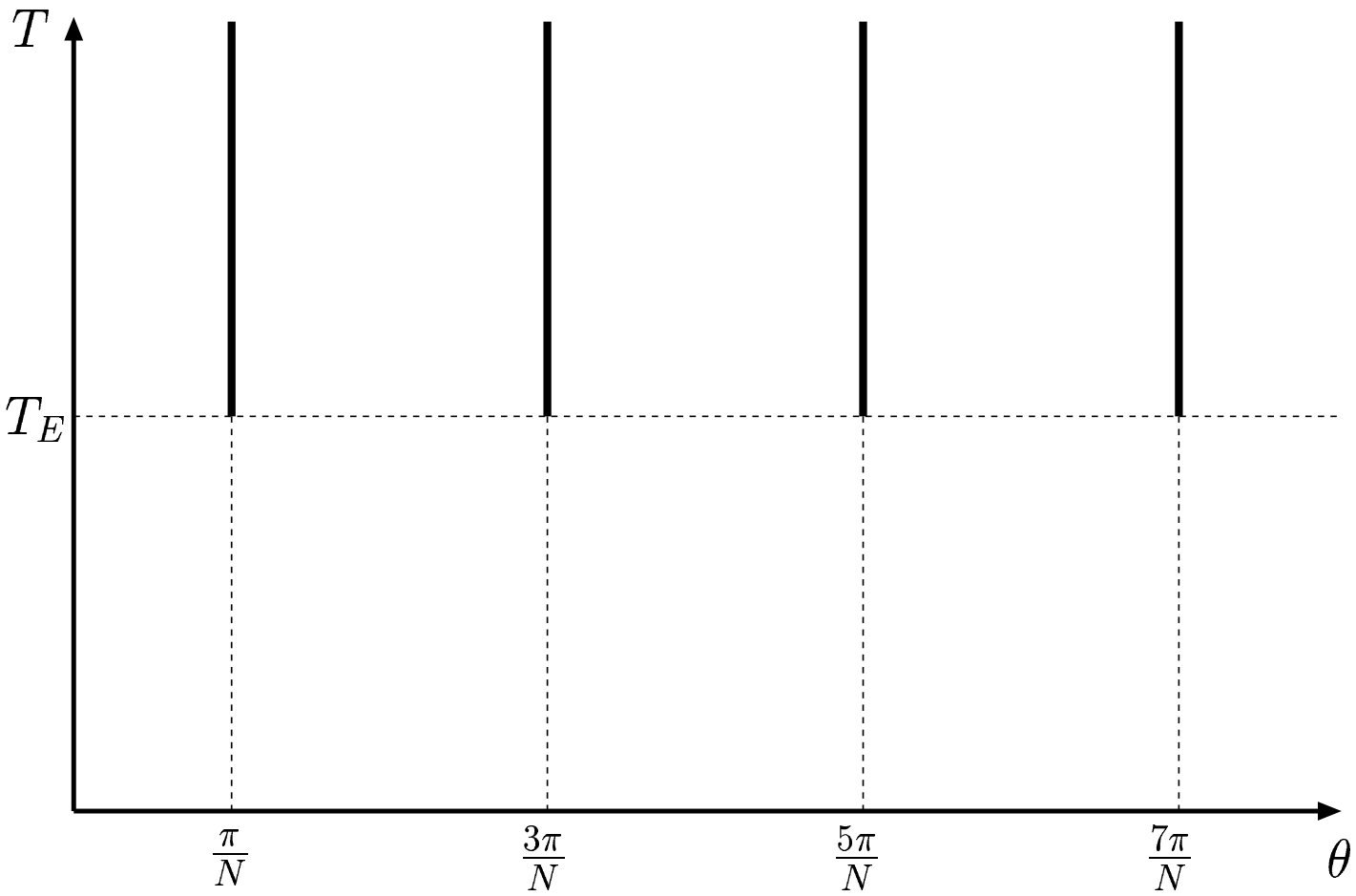

0.40
0.20
0.00
–0.20
–0.40
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
<L>
aμI
lattice 8^3x4
β=1.80
am=0.07
stat. 1K


lattice 8^3x4
β=1.80
am=0.07
stat. 1K
chiral condensate
aμ_I
0.12
0.11
0.10
0.09
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4

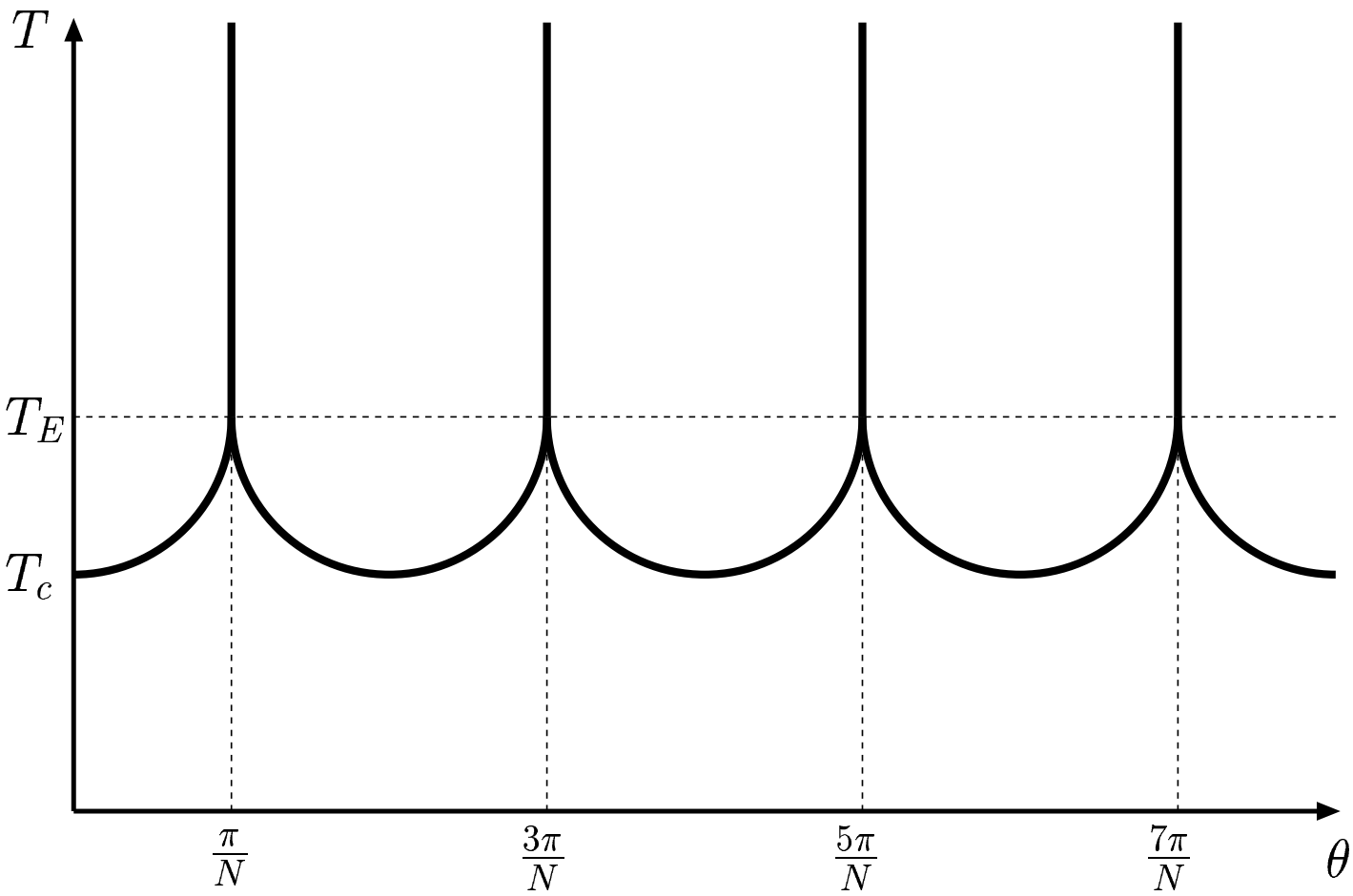

lattice 8^3x4
β=1.35
am=0.07
stat. 1K
<L>
aμ_I
0.20
0.10
0.00
−0.10
−0.20
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4


chiral condensate
0.38
0.36
0.34
0.32
0.30
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
aμI
lattice 8^3x4
β=1.35
am=0.07
stat. 1K

Phase diagram on the (T,θ)-plane



Phase diagram on the (T,θ)-plane



Outline

Description and state-of-the-art

- Strategy of the method of analytical continuation
  [M.G. Alford, A. Kapustin, F. Wilczek, 1999]
  [M.P. Lombardo, 2000]

  - determine $\langle \mathcal{O} \rangle$ for a set of value of imaginary chemical potential, $\mu = i\mu_I$
  - interpolate $\langle \mathcal{O} \rangle(\mu)$ with a polynomial:

$$\langle \mathcal{O} \rangle(\mu) = a_0 + a_2\mu^2 + a_4\mu^4 + a_6\mu^6 + O(\mu^8)$$

  - analytically continue to $\mu = \mu_R$ by the replacement $\mu^2 \to -\mu^2$

$$\langle \mathcal{O} \rangle(\mu) = a_0 - a_2\mu^2 + a_4\mu^4 - a_6\mu^6 + O(\mu^8)$$

- Applied in
  - SU(3), $n_f = 2$ [Ph. de Forcrand, O. Philipsen, 2002]
  - SU(3), $n_f = 3$ [Ph. de Forcrand, O. Philipsen, 2003]
  - SU(3), $n_f = 4$ [M. D'Elia, M.P. Lombardo, 2003]

Description and state-of-the-art

- Tested in
  - $3d$ SU(3) + adjoint Higgs model [A. Hart, M. Laine, O. Philipsen, 2001]
  - SU(2), $n_f = 8$ [P. Giudice, A.P., 2004]
  - $3d$ 3-state Potts model [S. Kim et al., 2005]

- So far, the method has been applied using polynomials as interpolating functions

- Here, we propose and test an approach based on the use of ratio of polynomials

Description and state-of-the-art

- Tested in
  - $3d$ SU(3) + adjoint Higgs model [A. Hart, M. Laine, O. Philipsen, 2001]
  - SU(2), $n_f = 8$ [P. Giudice, A.P., 2004]
  - $3d$ 3-state Potts model [S. Kim et al., 2005]

- So far, the method has been applied using polynomials as interpolating functions

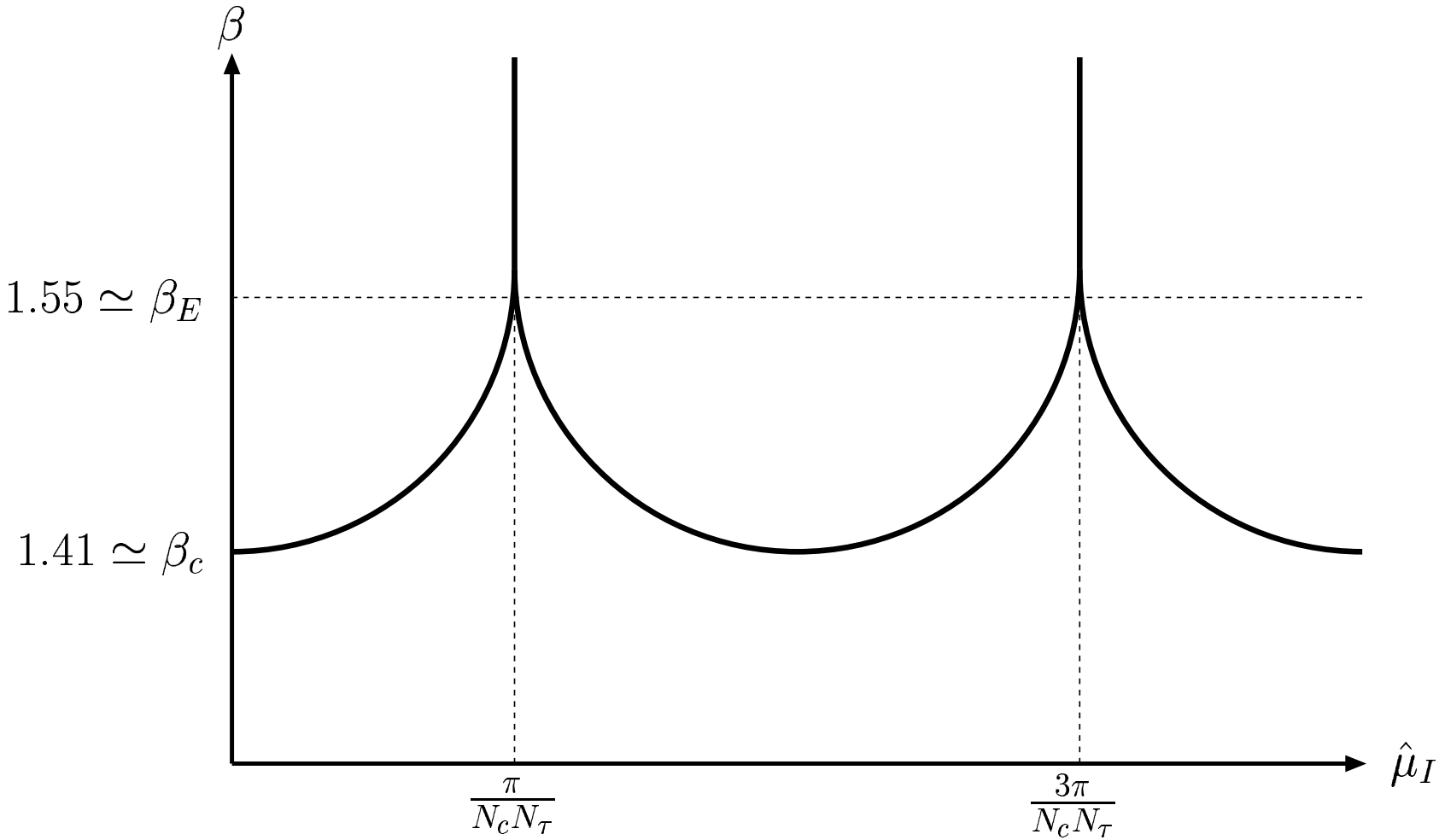
- Here, we propose and test an approach based on the use of ratio of polynomials

Outline

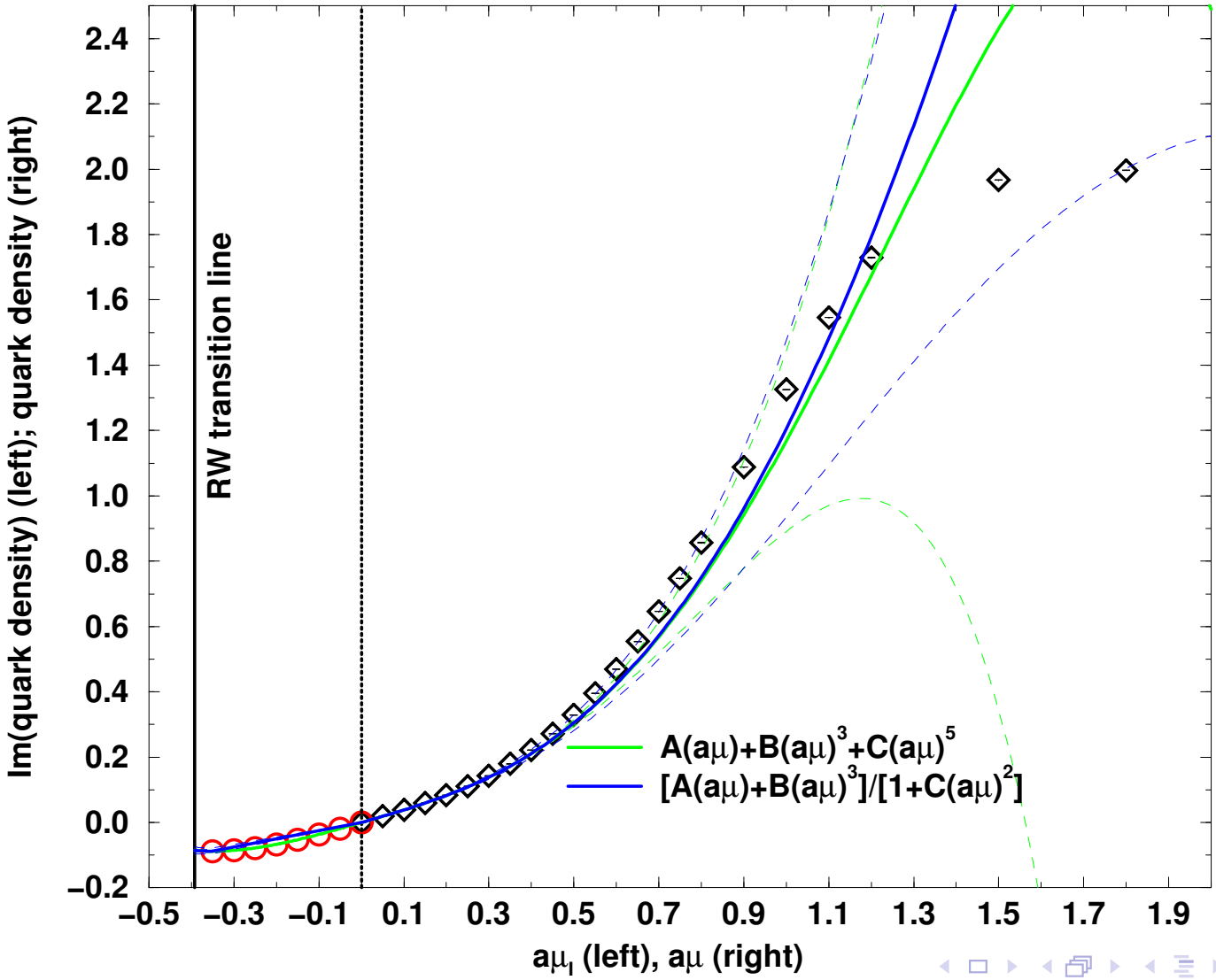
Numerical results - Details on the lattice simulations

- SU(2) gauge theory with n_f=8 staggered fermions, fermion mass am=0.07, on a $16^3 \times 4$ lattice
- hybrid Monte Carlo algorithm, with dt=0.01
- observables (statistics 1000-5000):
  - chiral condensate $\langle \bar{\psi}\psi \rangle$
  - Polyakov loop
  - fermion number density
- simulations on the APEmille crate in Bari and on the recently installed computer facilities at the INFN APEnext Computing Center
- preliminary results at $\beta = 1.90 > \beta_E$, for which the window in μ_I free of non-analyticities is the largest possible

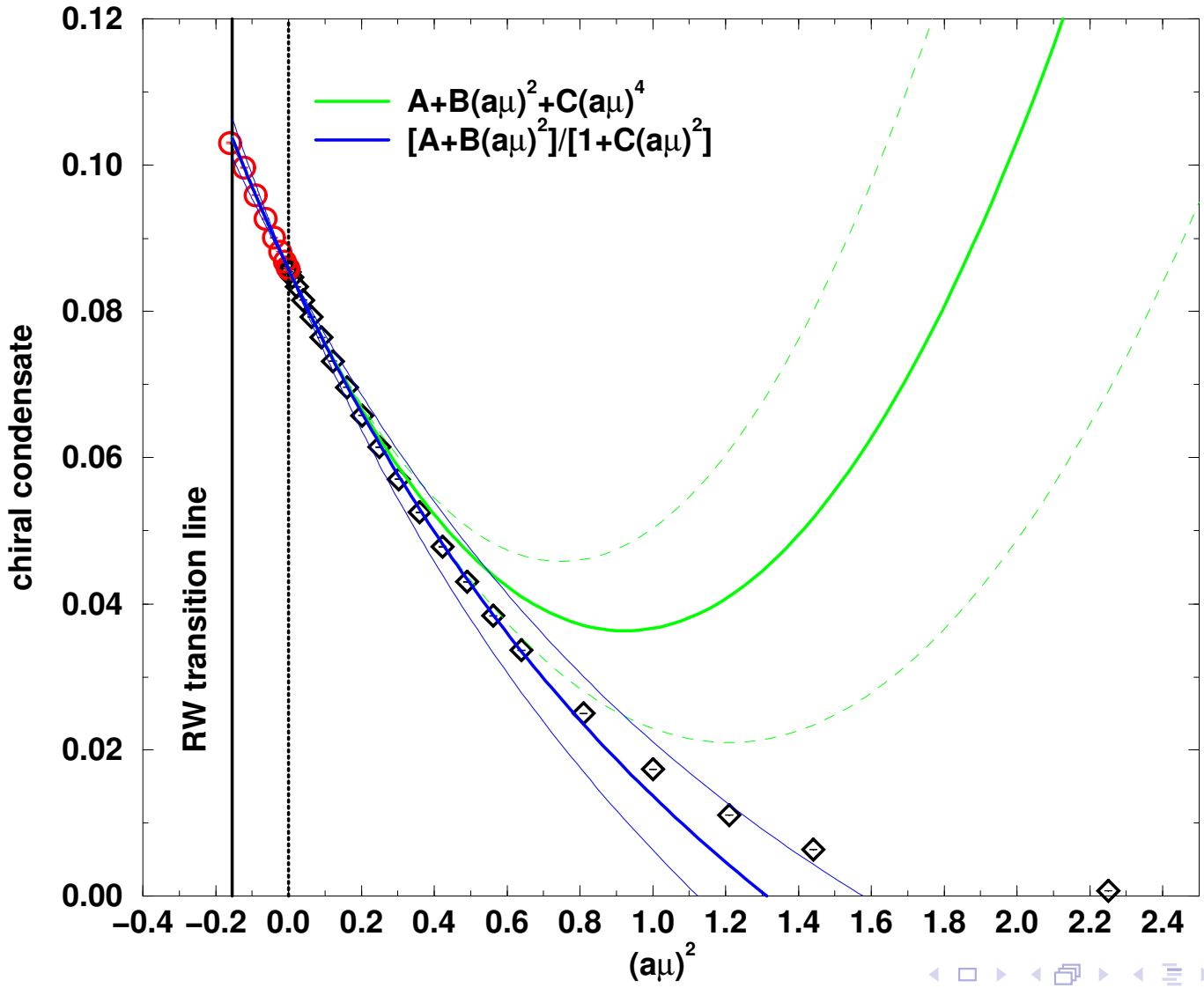
Numerical results



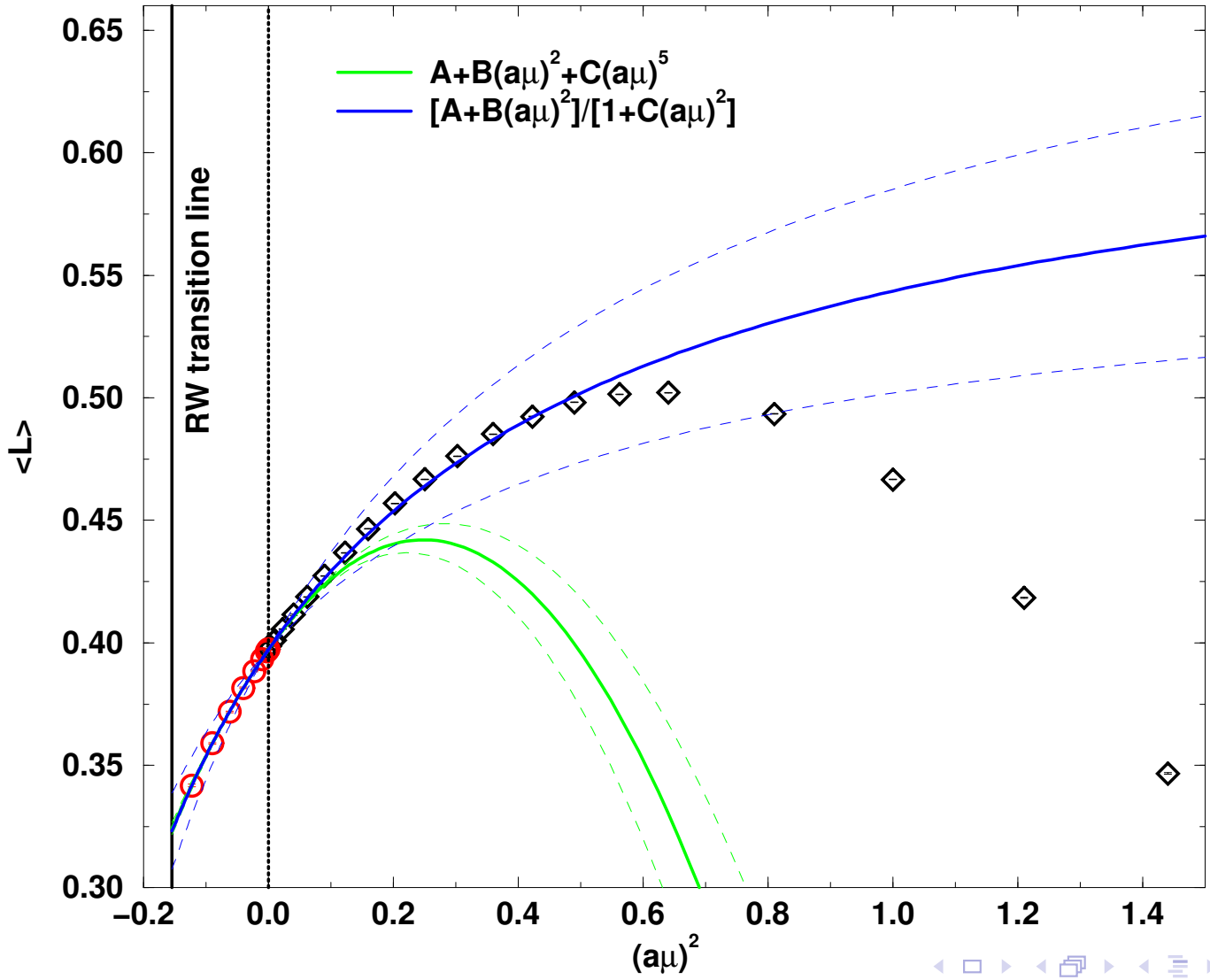
Numerical results - Fermion number density



Numerical results - Chiral condensate



Numerical results - Polyakov loop



Conclusions

- We have verified by comparison with direct Monte Carlo determinations at real chemical potential in 2-color QCD that the method of analytical continuation considerably improves if ratio of polynomials is used as interpolating function instead of truncated Taylor series.
- In the case of the Polyakov loop and of the chiral condensate an interpolation of numerical data at imaginary chemical potential over the window permitted by Roberge-Weiss singularities allows an extrapolation to real values of the chemical potential over a much larger region.
  Deviations at very large values of the chemical potential could be due to unphysical saturation of the fermionic density ("Pauli blocking").
- The presence of the Roberge-Weiss transition has no influence on the analyticity of the partition function at real values of μ.
- Our method looks very promising in view of applications to real QCD.

Outlook

- It is in progress the test to other values of the gauge coupling, in particular to those values for which non-analyticities (different from the first order Roberge-Weiss) are expected for real or imaginary chemical potential
- Forthcoming applications of this method
  - analytical continuation of the critical line in 2-color QCD
  - SU(3) with finite isospin density