The fractal self-similar-Borel algorithm

By

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The fractal self-similar-Borel Cont'd

Outline:

- THE FRACTAL SELF-SIMILAR METHOD
- Applications to non-Hermitian field theory
- The kleinert Algorithm for Borel Resummation
- THE FRACTAL SELF-SIMILAR-BOREL Algorithm
- Applications:
  1. Critical Exponents of 3-dimensional XY model
  2. Critical Coupling of The ϕ^4_{1+1} theory from the resummation of the effective potential

- Conclusions

THE FRACTAL SELF-SIMILAR METHOD

(Due to Yukalov et.al)

- Consider the series given by

$$p_k = \sum_{k=0}^{k} a_n x^n.$$

- Applying the fractal transform we get

$$P_k(x, s) = x^s P_k(x) = \sum_{n=0}^{k} a_n x^{s+n}.$$

THE FRACTAL SELF-SIMILAR Cont'd

- It is fractal because it satisfies the relation

$$\frac{P(\lambda x, s)}{p(\lambda x)} = \lambda^s \frac{P(x, s)}{p(\lambda x)}$$

- Define the initial approximation

$$P_0(x, s) = a_0 x^s = f.$$

- Solving for *x* we get

$$x(f, s) = \left(\frac{f}{a_0}\right)^{\frac{1}{s}}.$$

THE FRACTAL SELF-SIMILAR Cont'd

- **Then The Cascade y_k is given by:**

$$y_k(f,s) = P_k(x(f,s),s) = \sum_{n=0}^{k} a_n \left(\frac{f}{a_0}\right)^{\frac{n}{s+1}}.$$

The cascade velocity is given by

$$y_k(f,s) - y_{k-1}(f,s) = \sum_{n=0}^{k} a_n \left(\frac{f}{a_0}\right)^{\frac{n}{s+1}} - \sum_{n=0}^{k-1} a_n \left(\frac{f}{a_0}\right)^{\frac{n}{s+1}}$$
$$= a_k \left(\frac{f}{a_0}\right)^{\frac{n}{s+1}}.$$

THE FRACTAL SELF-SIMILAR Cont'd

- The regime of the self-similar renormalization is to consider the passage from one approximation to another as a motion with respect to the approximation number k = 0, 1,2,
- The trajectory y_k (f, s) of this dynamical system is bijective to the approximation sequence P_k (x, s).
- The attracting fixed point of the cascade trajectory is, by construction, bijective to the limit of the approximation sequence P_k (x, s), that is, it corresponds to the sought

  function.

THE FRACTAL SELF-SIMILAR Cont'd

- One can deal with continuous time *t* rather than the discrete time *k.*
- The evolution equation for the flow reads

$$\frac{\partial}{\partial t} y(t, f, s) = v(y(t, f, s))$$

- Accordingly, the evolution integral is

$$\int_{P_k}^{P_{k+1}^*} \frac{df}{v_{k+1}(f, s)} = t_k^*.$$

THE FRACTAL SELF-SIMILAR Cont'd

- Thus, the self-similar approximation is given by

$$p_k^* = p_{k-1}(x)\left(1 - \frac{k a_k}{s a_0^{1+\frac{k}{s}}} x^k p_{k-1}^{\frac{k}{s}}(x)\right)^{\frac{-s}{k}},$$

- The applicability of the method is governed by the stabilizers

$$\mu_k(f) = \frac{\partial}{\partial f} y_k(f, s),$$

Or their images

$$m_k(x, s) = \mu_k(P_0(x, s), s)$$

THE FRACTAL SELF-SIMILAR Cont'd

- The stability condition is given by

$$|m_k(x, s)| < 1$$

- For the series given above we have

$$m_k(x, s) = \sum_{n=0}^{k} \frac{a_n}{a_0}\left(1 + \frac{n}{s}\right) x^n$$

- For *k* = 3, the stabilizers are given by

$$m_k(x, s) = \frac{xa_1 + 2x^2a_2 + 3x^3a_3}{sa_0} + \frac{a_0 + xa_1 + x^2a_2 + x^3a_3}{a_0}$$

THE FRACTAL SELF-SIMILAR Cont'd

- The most stable aproximant is obtained if $m_k\,(x, s) = 0$, or

$$s = -\frac{xa_1 + 2x^2a_2 + 3x^3a_3}{a_0 + xa_1 + x^2a_2 + x^3a_3}$$

- Otherwise, the minimum occurs at s=∞. Therefore, the minimum is given by

$$\left|m_k\left(x, s\right)\right|_{s\to\infty} = \left|\frac{a_0 + xa_1 + x^2a_2 + x^3a_3}{a_0}\right|$$

THE FRACTAL SELF-SIMILAR Cont'd

- If it happens that all m_k's are less than 1 for all s's are ∞, then the resummed series is given by the bootstrap formula

$$p_k^* = a_0 \exp\left(\frac{a_1}{a_0} x \exp\left(\frac{a_2}{a_1} x \exp\left(\frac{a_3}{a_2} x \exp\left(\frac{a_4}{a_3} x \exp\left(\frac{a_5}{a_4} x \ldots\ldots \exp\left(\frac{a_n}{a_{n-1}} x\right)\right)\right)\right)\right)\right)$$

Applications to an example with known exact result for comparison

- Consider the Lambert *W* function defined by

$$W(x)\exp(W(x)) = x.$$

- The series expansion of *W* (1 + *x*) is

$$W(1+x) \approx W(1) + \frac{W(1)}{1+W(1)}x + \left(-\frac{1}{2}(W(1))^2\frac{2+W(1)}{(1+W(1))^3}\right)x^2 + \left(\frac{1}{6}(W(1))^3\frac{9+8W(1)+2(W(1))^2}{(1+W(1))^5}\right)x^3 + O(x^4).$$

Applications to an example with known exact result cont'd

- At $x = 3$, $W(1 + x) = 1.2022$ and the perturbative result (up $x3$) is 1. 918 9. The error percent is 59.616%.

- Let us apply the transformation,:

$$Y(W(1+x)) = W(1+x) + c$$

where c is used as a control function too.

- Apply the fractal self-similar method to $Y(W(1 + x))$ and find c which makes all the $|mk(x, s)|_{s \to \infty}$ less than one and then apply Y^{-1} to the obtained bootstrap formula we get the result $W(1 + x) \approx 1.1798$ with the error percent 2.0697%.

Applications to a non-Hermitian Field theory model

- Consider the Lagrangian density:

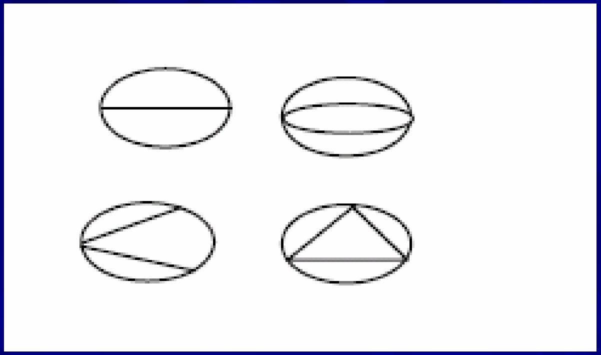
$$L = \frac{1}{2}\left((\partial\phi)^2 - m^2\phi^2\right) + \frac{g}{4}\phi^4$$

- This model is not Borel Summable due to the existence of classical soliton solution
- In the equivalent quasi-field theory, the interaction term is

$$-\frac{g}{4}\psi^4 - gB\psi^4$$

- Up to g^3, we have the Feynman diagrams (non-cactus) shown in Fig.1.

Applications to a non-Hermitian Field theory model



- Accordingly, the perturbation corrections to the Effective Potential are

$$\frac{8\pi E(b,t,G)}{m^2} = t - \ln t + b^2 - 1 - G\left(\frac{1}{4}b^4 + \frac{3}{4}\ln^2 t - \frac{3}{2}b^2 \ln t\right) + G^2\left(-\frac{3.155}{t} - 3.515\frac{b^2}{t}\right) - G^3\left(\frac{4.057}{t^2} + 9.918\frac{b^2}{t^2}\right)$$

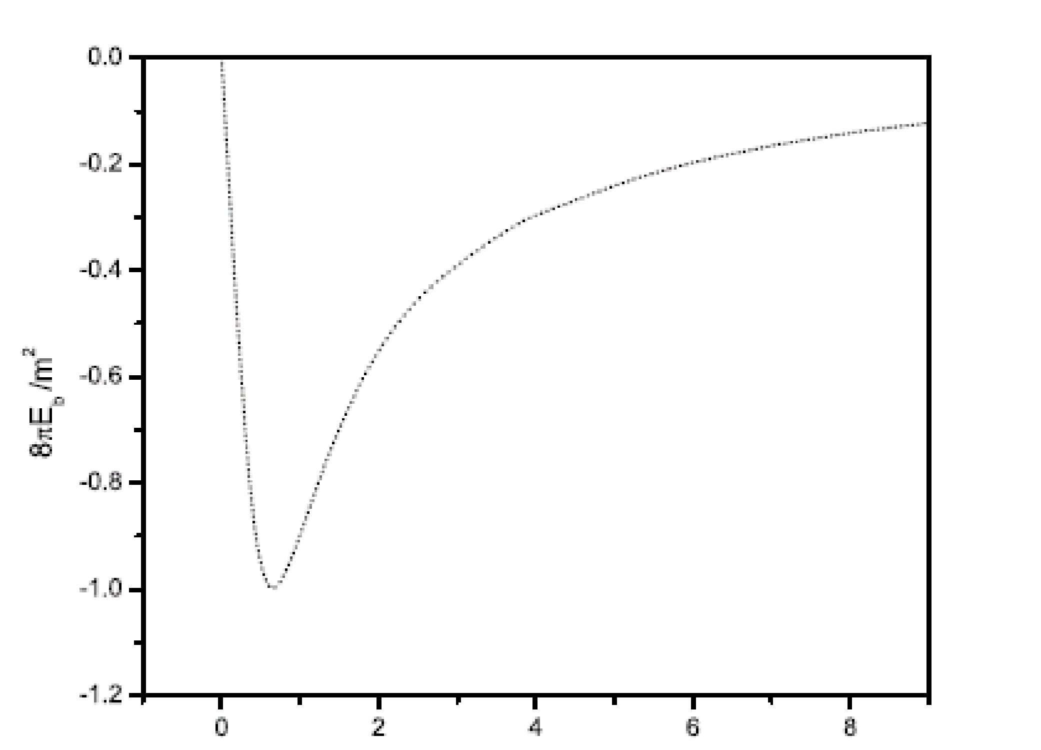
Applications to a non-Hermitian cont'd

- To keep the equivalence, we use the fact that the bare parameters are independent of the scale *t.* Accordingly, we obtain the result

$$\frac{8\pi E(t,b,G)}{m^2} = t - \ln t + b^2 - 1 - G\left(\frac{1}{4}b^4 + \frac{3}{4}\ln^2 t - \frac{3}{2}b^2\ln t\right)$$
$$+ G^2\left(-3.155\left(\frac{1}{t} - 1\right) - 3.515b^2\left(\frac{1}{t} - 1\right)\right)$$
$$- G^3\left(4.057\left(\frac{1}{t^2} - 1\right) + 9.918b^2\left(\frac{1}{t^2} - 1\right)\right)$$

Applications to a non-Hermitian cont'd

The resummed Vacuum energy E_b as a function of the coupling G



E_b is the resummation of the field dependent terms only.

Applications to a non-Hermitian cont'd

- The resummed E_b agrees qualitatively with pervious results concerning bound states

(see Carl M. Bender, Stefan Boettcher, H. F. Jones, Peter N. Meisinger, and Mehmet Simsek,Phys.Lett.A291, 197 (2001).

The kleinert Algorithm for Borel Resummation

- Suppose that the asymptotically onverging series to be summed is given by

$$E(G) = \sum_k Z_k G^k$$

- the large order behavior of the series is known to be

$$Z_k \to (-1)^k k! k^\delta \sigma^k \left(\gamma_0 + \frac{\gamma_1}{k} + \frac{\gamma_2}{k^2} + \ldots\ldots\right), \text{ as } k \to \infty,$$

- The strong coupling behavior of *E*(*G*) is given by

$$E(G) \to c_s G^\alpha, \text{ as } G \to \infty$$

H_p^c

The kleinert Algorithm Cont'd

- After the change of basis, $E(G)$ is written as

$$E(G) = \sum_{p=0}^{\infty} a_p I_p(G)$$

- The functions $I_p(G)$ are chosen to have the Borel representation:

$$I_p(G) = \int_0^{\infty} dt e^{-t} t^c H_p^c(Gt)$$

- where H_p^c are constrained in such a way that $I_p(G)$ satisfies both large order and stron coupling behaviors.

The kleinert Algorithm Cont'd

- After some algebraic steps one find that

$$I_p(G) = \int_0^\infty dt \frac{e^{-t}t^c}{\Gamma(c+1)} \frac{(\sigma G t)^p}{4^p} {}_2F_1(p-\alpha, p-\alpha+\frac{1}{2}; 2(p-\alpha)+1; -\sigma G t)$$

$$= \int_0^\infty dt \frac{e^{-t}t^c}{\Gamma(c+1)} (\sigma G t)^p 4^{-\alpha} \left(1+\sqrt{1+\sigma G t}\right)^{-2(p-\alpha)},$$

will do the Job.

- a_p can be found to be

$$a_p = \sum_{k=0}^{p} (-1)^{p-k} \frac{Z_k}{(c+1)_k} \left(\frac{4}{\sigma}\right)^k \binom{-2(k-\alpha)}{p-k}$$

THE FRACTAL SELF-SIMILAR-BOREL Algorithm

- With no loss of generality, let us write E(G,x) as:

$$E(G,x) = \sum_{p=0}^{N} a_p I_p x^p \equiv D_p x^p$$

This series coincides with ours for x=1.

- To accelerate the convergence of the above series, we apply the fractal self-similar to the above series and at the end of the day we get back to x=1.

Applications

1. Critical Exponents of 3-dimensional XY model.

- **Consider the series (Erratum-ibid. B319 (1993) 545 Phys.Lett. B272 (1991) 39-44)**

$$
\begin{aligned}
1/\nu(\epsilon) \;=\; & 2+\frac{(n+2)\epsilon}{n+8}\Big\{-1-\frac{\epsilon}{2(n+8)^2}(13n+44) \\
& +\frac{\epsilon^2}{8(n+8)^4}\,[3n^3-452n^2-2672n-5312 \\
& \qquad +\zeta(3)(n+8)\cdot 96(5n+22)] \\
& +\frac{\epsilon^3}{32(n+8)^6}\,[3n^5+398n^4-12900n^3-81552n^2-219968n-357120 \\
& \qquad +\zeta(3)(n+8)\cdot 16(3n^4-194n^3+148n^2+9472n+19488) \\
& \qquad +\zeta(4)(n+8)^3\cdot 288(5n+22) \\
& \qquad -\zeta(5)(n+8)^2\cdot 1280(2n^2+55n+186)] \\
& +\frac{\epsilon^4}{128(n+8)^8}\,[3n^7-1198n^6-27484n^5-1055344n^4 \\
& \qquad -5242112n^3-5256704n^2+6999040n-626688 \\
& \qquad -\zeta(3)(n+8)\cdot 16(13n^6-310n^5+19004n^4+102400n^3 \\
& \qquad\qquad -381536n^2-2792576n-4240640) \\
& \qquad -\zeta^2(3)(n+8)^2\cdot 1024(2n^4+18n^3+981n^2+6994n+11688) \\
& \qquad +\zeta(4)(n+8)^3\cdot 48(3n^4-194n^3+148n^2+9472n+19488) \\
& \qquad +\zeta(5)(n+8)^2\cdot 256(155n^4+3026n^3+989n^2-66018n-130608) \\
& \qquad -\zeta(6)(n+8)^4\cdot 6400(2n^2+55n+186) \\
& \qquad +\zeta(7)(n+8)^3\cdot 56448(14n^2+189n+526)]\Big\}
\end{aligned}
$$

Critical Exponents cont'd

Up to ε^2	Borel result ν=0.65413 (α=0.03761)
Up to ε^3	ν=0.65879 (α= 0.02363)
Up to ε^4	ν=0.66527 (α= 0.00419)
Up to ε^5	ν=0.66604 (α= 0.00188)

- **The best experimental result for** α is -0.0127±0.0003

A. Lipa et.al, Phys. Rev. B 68, 174518 (2003)

Critical Exponents cont'd

Up to ε^5	**THE FRACTAL SELF-SIMILAR-BOREL algorithm** $\nu=0.67079$ ($\alpha=-0.01237$)

Critical Coupling of The ϕ^4_{1+1} theory

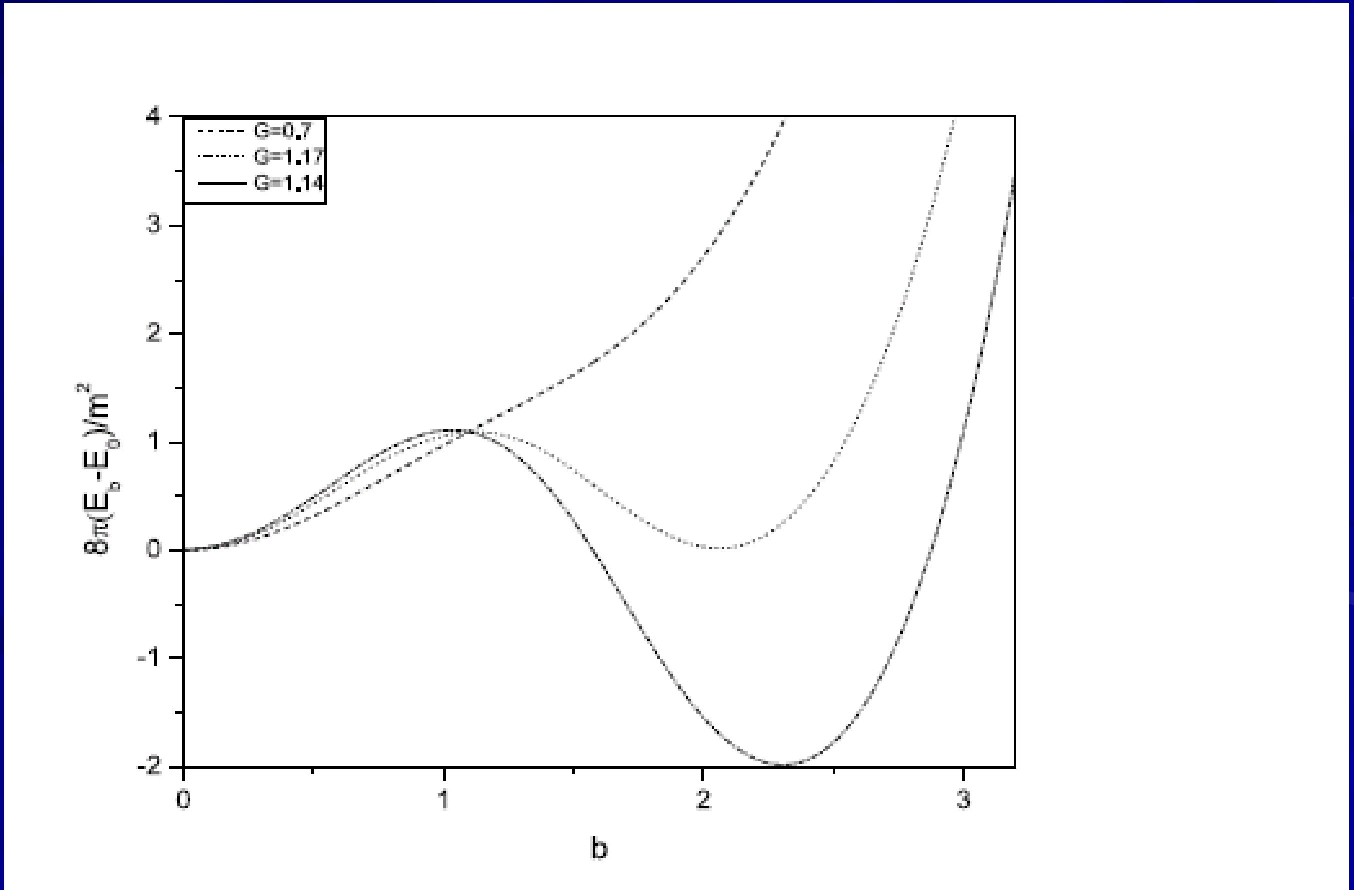
- For the effective potential of the ϕ^4_{1+1} , the perturbation series up to G^3 is of the form:

$$\frac{8\pi E(t,b,G)}{m^2} = t - \ln t + b^2 - 1 + G\left(\frac{1}{4}b^4 + \frac{3}{4}\ln^2 t - \frac{3}{2}b^2 \ln t\right) + G^2\left(-\frac{3.155}{t} - 3.515\frac{b^2}{t}\right) + G^3\left(\frac{4.057}{t^2} + 9.918\frac{b^2}{t^2}\right)$$

- The critical coupling calculated from the perturbative series is G_c=1.17.
- The critical coupling calculated from the Borel resummation is G_c=1.

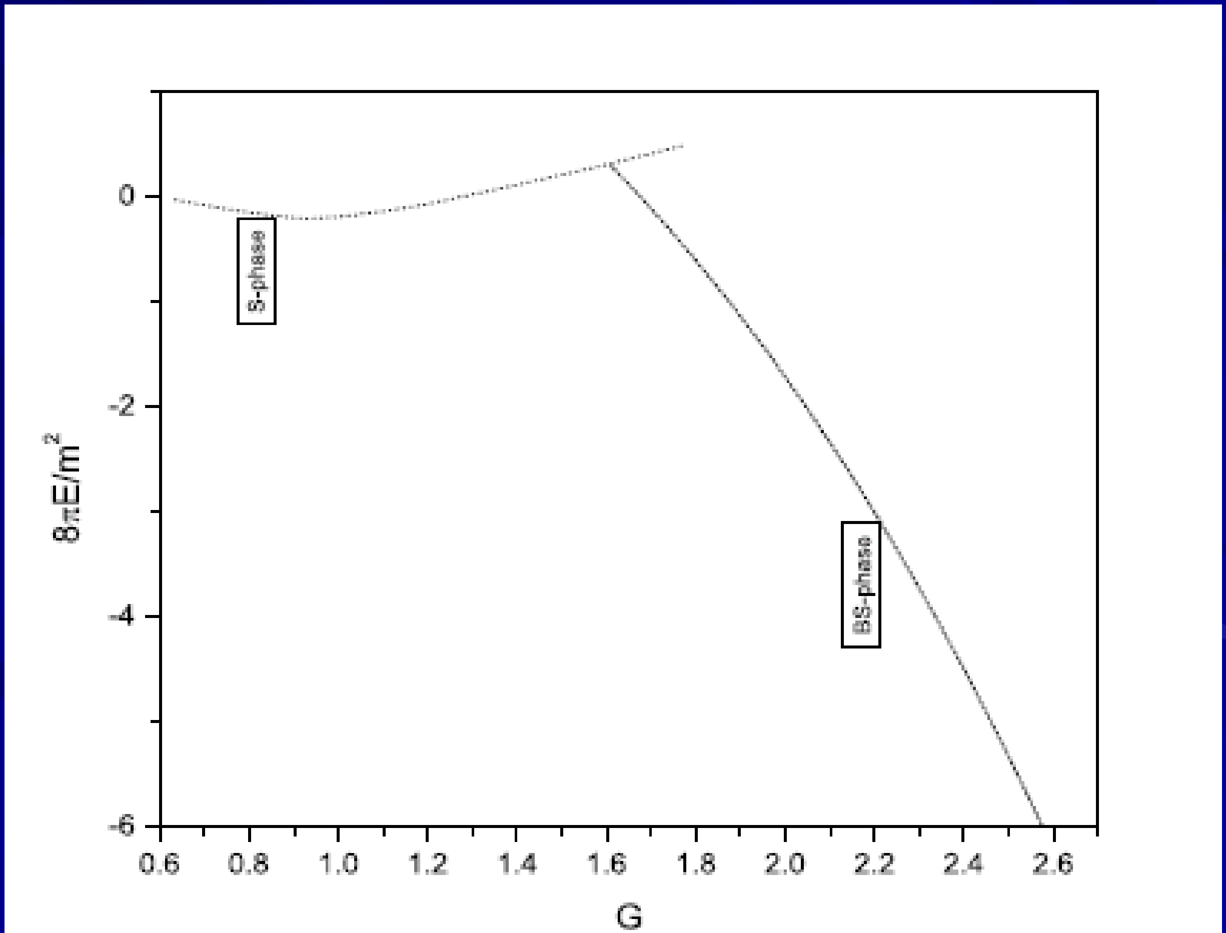
Critical Coupling Cont'd

- The perturbative effective potential up to G^3 for $G = 0.7$, $G = 1.17$ and $G = 1.4$.



Critical Coupling Cont'd

- The effective potential as a function of the coupling *G* for both S and BS phases obtained from the fractal self-similar-Borel Method. The obtained G_c=1.6259 (Lattice is 1.625).



Conclusions

- The Borel method needs many terms of perturbation series to achieve reliable result.
- Borel method supplemented by the self-similar method accelerates the convergence of the resummed result and give reliable results for the critical coupling of the ϕ^4_{1+1} theory even with the input perturbation series up to G^3 only.

Conclusions

- For the critical exponents of the XY model our algorithm is consistent with the best experimental result obtained so far for the α exponent of the specific-heat peak in superfluid helium, found in a satellite experiment with a temperature resolution of nanoKelvin.