TRIMERIC CHAIN OF BOSE-EINSTEIN CONDENSATES: PARAMETER DEPENDENCE AND CHAOS ONSET

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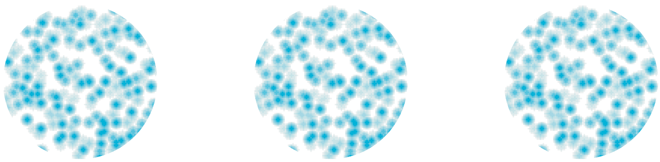


- arrays of B-E condensates
- many body $\longrightarrow$ (lattice) mean field
- (mean field) trimer dynamics
- stability diagrams
- simulations (possible experiments involving macroscopic effects)
- conclusions & perspectives.

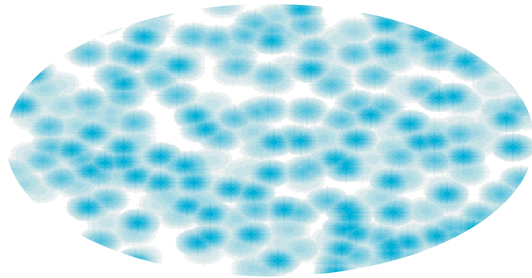
ARRAYS OF INTERACTING BOSE-EINSTEIN CONDENSATES

gas of alkali atoms

N bosons

Low Temperatures

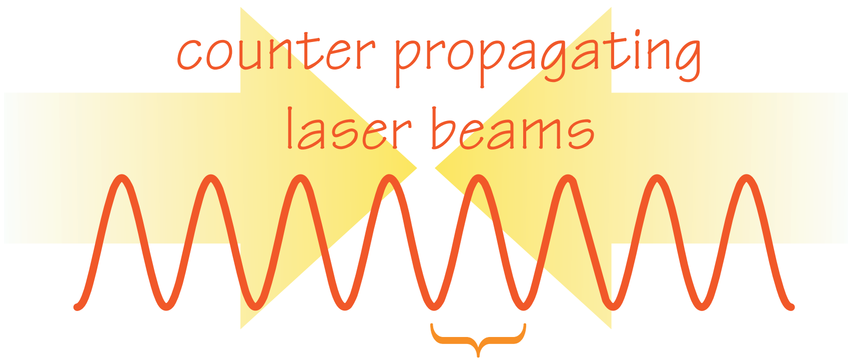
$(T < T_c \sim 500 \text{ nK})$



magnetic harmonic trap

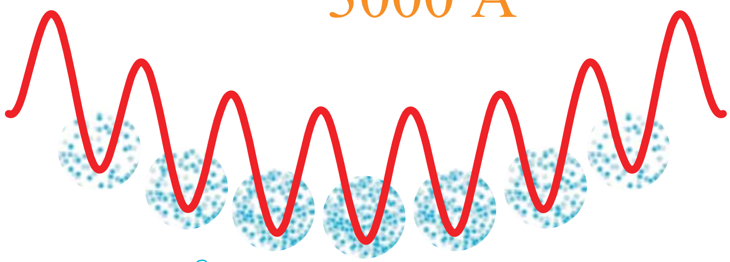
optical lattice

sinusoidal effective potential



trapping potential

array of weakly interacting B-E Condensates



Up to 10^2 condensates

average condensate population: 10^6-10^1

Anderson & Kasevich, Macroscopic Quantum Interference from Atomic Tunneling Arrays, *Science* **282**, 1686 (1998).

Cataliotti, Burger, Fort, Maddaloni, Minardi, Trombettoni, Smerzi & Inguscio, Josephson Junction Arrays with Bose-Einstein Condensates, *Science* **293**, 843 (2001)

Cristiani, Morsch, Müller, Ciampini & Arimondo, Experimental Properties of Bose-Einstein Condensates in 1D Optical Lattices: Bloch Oscillations, Landau-Zener Tunneling and Mean Field Effects, **cond-mat/0202053**

MANY-BODY HAMILTONIAN

trapping potential

$$\hat{H} = \int d\mathbf{r}\,\hat{\psi}^{+}(\mathbf{r})\left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}\right]\hat{\psi}(\mathbf{r})$$

$$+\int d\mathbf{r}\,d\mathbf{r}'\,\hat{\psi}^{+}(\mathbf{r})\,\hat{\psi}^{+}(\mathbf{r}')\,g\,\delta(\mathbf{r}-\mathbf{r}')\,\hat{\psi}(\mathbf{r}')\,\hat{\psi}(\mathbf{r})$$

interaction potential

boson field creation & annihilation operators

elastic s-wave binary collisions

$$g = \frac{4\pi\hbar^2 a}{m}$$

s-wave scattering length

low temperature

$$\lambda(T) = \sqrt{\frac{2\pi\hbar^2}{m k_B T}} \gg a$$

low density

$$\rho a^3 \ll 1$$

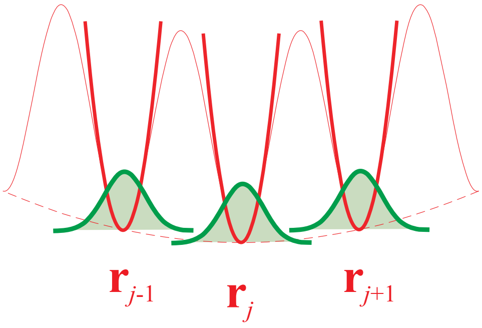
$$a \ll \ell = \rho^{-1/3} \approx \lambda(T_c) \ll a_{\text{ho}} \ll \sigma_T$$

5-10 nm　　200 nm　　$\sim 10^3$ nm　　$\sim 10^4$ nm

Parkins & Walls, The physics of trapped dilute-gas Bose-Einstein Condensates, *Phys. Rep.* 303, 1 (1998).

Dalfovo, Giorgini, Pitaevskii & Stringari, Theory of Bose-Einstein Condensation in trapped gases, *Rev. Mod. Phys.* **71**, 463 (1999).

SPACE-MODE APPROXIMATION



$V_j(\mathbf{r})$ parabolic approximation of $V_{ext}(\mathbf{r})$ at $\mathbf{r}_j$

$\mathbf{r}_j$ location of the j-th local minimum of $V_{ext}(\mathbf{r})$ (j-th lattice site)

$u_j(\mathbf{r})$ single particle ground-state mode of $V_j(\mathbf{r})$.

$$\hat{\psi}(\mathbf{r},t) \approx \sum_j u_j(\mathbf{r})\, \hat{a}_j(t)$$

boson operator at lattice site j

$$\hat{n}_j = \hat{a}_j^+ \hat{a}_j$$

number operator at lattice site j.

$$\int d\mathbf{r}\, u_j^*(\mathbf{r})\, u_k(\mathbf{r}) \approx \delta_{jk} \quad \Rightarrow \quad \left[\hat{a}_j(t), \hat{a}_k^+(t)\right] = \delta_{jk}$$

BOSE-HUBBARD HAMILTONIAN

$$\hat{H}^{\mathrm{BH}} = \sum_j \left[U\, \hat{n}_j \left(\hat{n}_j - 1\right) - v_j\, \hat{n}_j\right] - \frac{T}{2} \sum_{\langle j,k \rangle} \left(\hat{a}_j^+ \hat{a}_k + \hat{a}_k^+ \hat{a}_j\right)$$

overlap integrals

interatomic scattering

energy offset of site j.

hopping amplitude

$$v_j \approx V_{\mathrm{h.trap}}(\mathbf{r}_j)$$

Jaksch, Bruder, Cirac, Gardiner & Zoller, Cold Bosonic Atoms in Optical Lattices, *Phys. Rev. Lett.* **81**, 3108 (1998).

MEAN FIELD PICTURE: TIME DEPENDENT VARIATIONAL PRINCIPLE

$$|\Psi\rangle = e^{iS(t)}|Z\rangle$$ macroscopic trial state $$|Z\rangle = \otimes_j |z_j\rangle$$

$$\langle z_j|\hat{a}_j|z_j\rangle = z_j$$ Glauber coherent states $$|z_j\rangle = e^{-\frac{1}{2}|z_j|^2}\sum_{k=0}^{\infty}\frac{z_j^k}{k!}\left(a_j^+\right)^k|0\rangle$$

$$\langle\Psi|i\hbar\partial_t - \hat{H}^{\mathrm{BH}}|\Psi\rangle = 0$$

time evolution: weak form of the Schrödinger equation

$$S = \int dt\left[\langle Z|i\hbar\partial_t - \hat{H}^{\mathrm{BH}}|Z\rangle\right]$$

semiclassical effective action

TDVP $\delta S = 0$

$$i\hbar\dot{z}_j = \{z_j, \mathcal{H}\}$$

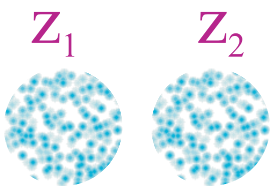
semiclassical effective Hamiltonian

$$\mathcal{H} = \sum_j\left(U|z_j|^2 - v_j\right)|z_j|^2 - \frac{T}{2}\sum_{\langle j,k\rangle}\left(z_j^* z_k + z_k^* z_j\right)$$

Zhang, Feng & Gilmore, Coherent states: Theory and some applications, Rev. *Mod. Phys*. **62**, 867 (1990).

Amico & Penna, Dynamical Mean Field Theory of the Bose-Hubbard Model, *Phys. Rev. Lett*. **80**, 2189 (1998)

THE DIMER

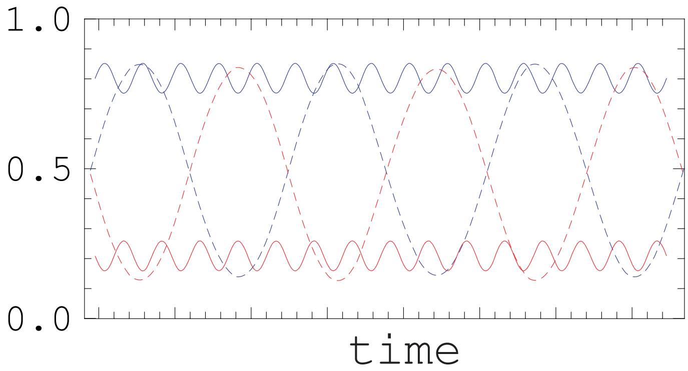


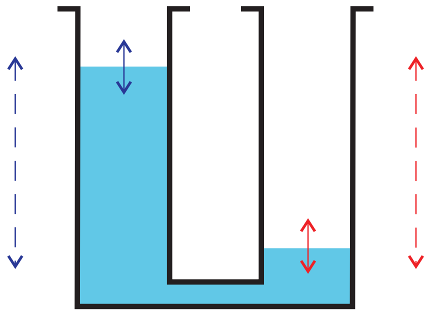
$$z_k = \sqrt{n_k}\,\exp\left(i\,\theta_k\right)$$

$$\mathcal{H} = U\left(|z_1|^4 + |z_2|^4\right) - v\left(|z_1|^2 + |z_2|^2\right) - T\left(z_1^* z_2 + z_1 z_2^*\right)$$

- Realized experimentally
- Widely studied theoretically (Q & MF picture)
- Many interesting macroscopic effects (due to nonlinearity) pointed out

macroscopic self-trapping





population imbalance

$$d \equiv \frac{|z_1|^2 - |z_1|^2}{N} = \frac{n_1 - n_2}{N}$$

$$\langle d \rangle \neq 0$$

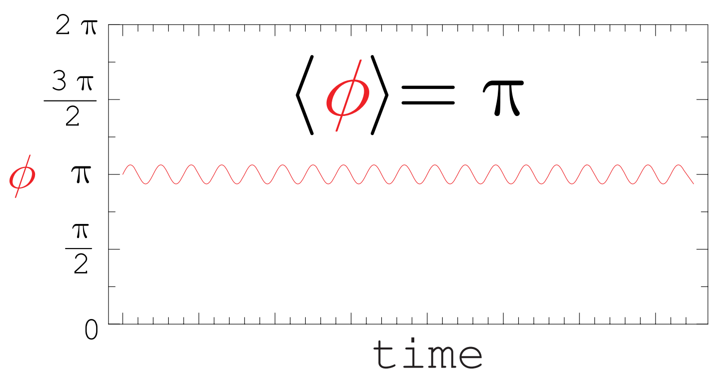
no population inversion

π-phase oscillations

$$\phi = \theta_1 - \theta_2$$

phase difference

phase coherence



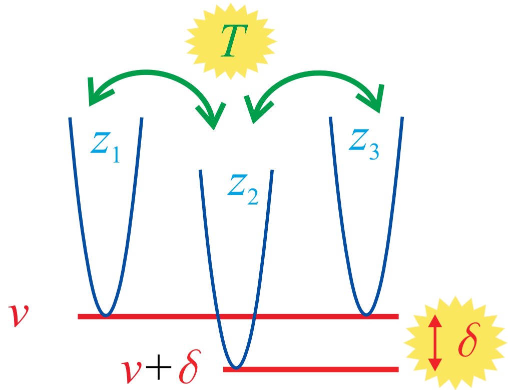
DIMER REFERENCES

Andrews & al. *Science* **275**, 673 (1997)
Aubry & al., *Phys. Rev. Lett.*, **76**, 1607 (1996)
Henning & al., *Phys. Rev.* A, **51**,2870 (1997)
Flach & al., *J. Phys.: Condens. Matter*, **9**, 7039 (1997)
Milburn & al. *Phys. Rev.* A, **55**, 4318 (1997)
Steel & al., *Phys. Rev.* A, **57**, 2920 (1998)
Corney & al., *Phys. Rev.* A, **58**, 2399 (1998)
Smerzi & al., *Phys. Rev.* A, **59**, 620 (1999)
Raghavan & al., *Phys. Rev.* A, **59**, 620 (1999)
Raghavan & al., *Phys. Rev.* A, **60**, R1787 (1999)
Franzosi & al. *Int. J. Mod. Phys.* B, **14**, 943 (2000)
Maddaloni & al. *Phys. Rev. Lett.*, **85**, 2413 (2000)
Franzosi & al. *Phys. Rev.* A **63** , 043609 (2001)
Vardi & al., *Phys. Rev. Lett.*, **86**, 568 (2001)
[...]

THE OPEN TRIMER

First (non-trivial) step beyond the (integrable) dimer

- Forthcoming practical realization.
- Non integrable
- Analytical tractabilty.



ADJUSTABLE

$$\tau = \frac{T}{U\,N} \qquad \nu = \frac{\delta}{U\,N} \qquad w = \frac{v}{U\,N}$$

rescaled Hamiltonian parameters

$$\zeta_j = \frac{z_j}{\sqrt{N}}$$ rescaled variables

total boson number

$$N = \sum_{j=1}^{3} \left| z_j \right|^2 = \sum_{j=1}^{3} n_j$$ number of bosons in the j-th well

$$\mathcal{H} = U\,N^2 \left\{ \sum_{j=1}^{3} \left(\left|\zeta_j\right|^4 - w \left|\zeta_j\right|^2 \right) - \nu \left|\zeta_2\right|^2 - \frac{\tau}{2} \left[(\zeta_1 + \zeta_3)\zeta_2^* + \text{c.c.} \right] \right\}$$

equations of motion

$$\begin{cases} i\hbar\,\dot{\zeta}_j = U\,N^2 \left[\left(2\left|\zeta_j\right|^2 - w \right)\zeta_j - \frac{\tau}{2}\zeta_2 \right] & j = 1,3 \\ i\hbar\,\dot{\zeta}_2 = U\,N^2 \left[\left(2\left|\zeta_2\right|^2 - w - \nu \right)\zeta_2 - \frac{\tau}{2}(\zeta_1 + \zeta_3) \right] & j = 2 \end{cases}$$

$$\sum_{j=1}^{3} \dot{n}_j = 0$$

3 REGIMES (initial conditions propagated in time)

$$\begin{cases} \zeta_1 = -\zeta_3 \neq 0 \\ \zeta_2 = 0 \end{cases}$$
depleted central well

$$\begin{cases} \zeta_1 = \zeta_3 \\ \zeta_j \neq 0 \quad \forall\, j \end{cases}$$
dimeric

$$\begin{cases} \zeta_1 \neq \zeta_3 \\ \zeta_j \neq 0 \quad \forall\, j \end{cases}$$
non-dimeric

Franzosi & Penna, Collective modes, chaotic behavior and self-trapping in the dynamics of three coupled Bose-Einstein condensates **cond-mat** 0203509.

FIXED POINTS & PERIODIC SOLUTIONS

Fixed points $0 = i\hbar \dot{z}_j = \{z_j, \mathcal{H} + \chi N\}$

Lagrange multiplier

$$0 = \dot{z}_j = \sqrt{N}\,\dot{\zeta}_j$$

selects the (conserved) value of N $\sum_{j=1}^{3} |z_j|^2 = N$

$$z_j = \sqrt{N}\zeta_j$$

$\zeta_j \in \mathbb{C}$ $\zeta_j \leftrightarrow \zeta_j \exp[i\,\Phi]$ $x_j \in \mathbb{R}$ $\sum_{j=1}^{3} |\zeta_j|^2 = \sum_{j=1}^{3} x_j^2 = 1$

$$\begin{cases} 0 = \left(2|x_j|^2 - \mu\right)x_j - \frac{\tau}{2}x_2 & j = 1,3 \\ 0 = \left(2|x_2|^2 - \mu - \nu\right)x_2 - \frac{\tau}{2}(x_1 + x_3) & j = 2 \end{cases} \qquad \mu = \frac{\nu + \chi}{U\,N}$$

two significant parameters, τ and ν (μ is dependent)

(x_1, x_2, x_3) ⟺ $\zeta_j(t) = x_j \exp\left[\frac{i}{\hbar}(\chi t + \varphi_0)\right]$

fixed point — collective periodic mode

depleted central well regime: $x_2 = 0,\ x_1 = -x_3 \neq 0$

$$(x_1, x_2, x_3) = \tfrac{1}{\sqrt{2}}(1, 0, -1)$$

just a single, parameter independent solution

FIXED POINTS: DIMERIC & NON-DIMERIC REGIMES

(x_1, x_2, x_3) → (R, θ) → $(R(\theta), \theta)$

constraint on number

fixed point equations

DIMERIC REGIME

$0 \neq x_2 \neq x_1 = x_3$

$$\begin{cases} x_1 = x_3 = \frac{R(\theta)}{\sqrt{2}} \cos\theta \\ x_2 = R(\theta) \sin\theta \end{cases}$$

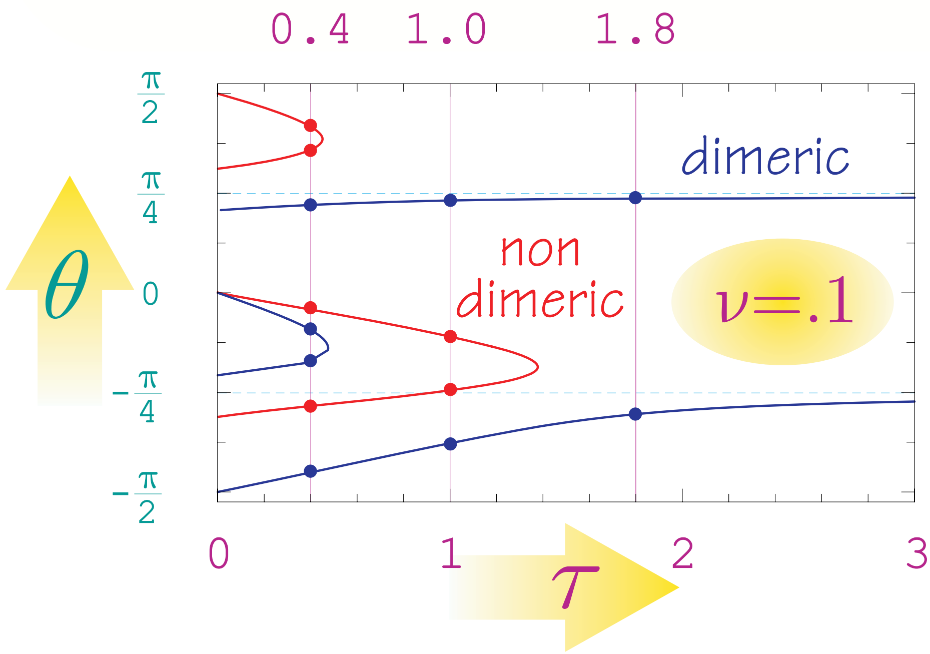
$$R(\theta) = 1$$

$$\mathcal{T} \equiv \tan(\theta) = \frac{x_2}{\sqrt{2}\, x_1}$$

for any pair of parameters (τ, ν)

$$\mathcal{P}_{\tau\nu}(\theta) \equiv a(\tau)\mathcal{T}^4 + b(\nu)\mathcal{T}^3 + c(\nu)\mathcal{T} + d(\tau)$$

graphical representation (stability diagrams)



$$\mathcal{P}_{\tau\nu}(\theta) = 0$$

$\tau - \theta$ plane

a one parameter (ν) family of curves for each regime

LOCAL CHARACTER AND LINEAR STABILITY OF THE FIXED POINTS:

very close to the fixed point

$$\zeta_j = x_j + q_j + i\,p_j$$

$$\mathbf{v} = (\mathbf{q}, \mathbf{p})$$

global phase symmetry $\Rightarrow x_j \in \mathbb{R}$

local character

$$\mathcal{H}(\zeta_j) \approx \mathcal{H}(x_j) + U\,N^2 \tau\, \mathbf{v}^t\, \mathcal{C}\, \mathbf{v}$$

$$\mathcal{C} = \begin{pmatrix} Q & 0 \\ 0 & P \end{pmatrix}$$

hessian

$\mathcal{C}$ eigenvalues (signature) $\{c_j\}$ → local character of the fixed point

$$\begin{cases} c_j < 0 \;\forall j & \text{max} \\ c_j > 0 \;\forall j & \text{min} \\ \text{otherwise} & \text{sdl} \end{cases}$$

linear stability

$$\dot{\mathbf{v}} = \{\mathbf{v}, \mathcal{H}\} = \tfrac{\tau}{2\hbar}\, U\, N\, \mathcal{S}\, \mathbf{v}$$

$$\mathcal{S} = \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix}$$

$\mathcal{S}$ eigenvalues $\{s_j\}$ → linear stability of the fixed point

$$\begin{cases} \mathrm{Re}(s_j) \le 0 \;\forall j & \text{stable} \\ \exists\, \mathrm{Re}(s_j) > 0 & \text{unstable} \end{cases}$$

$$\mathcal{S}^2 = -\begin{pmatrix} (QP)^t & 0 \\ 0 & QP \end{pmatrix}$$

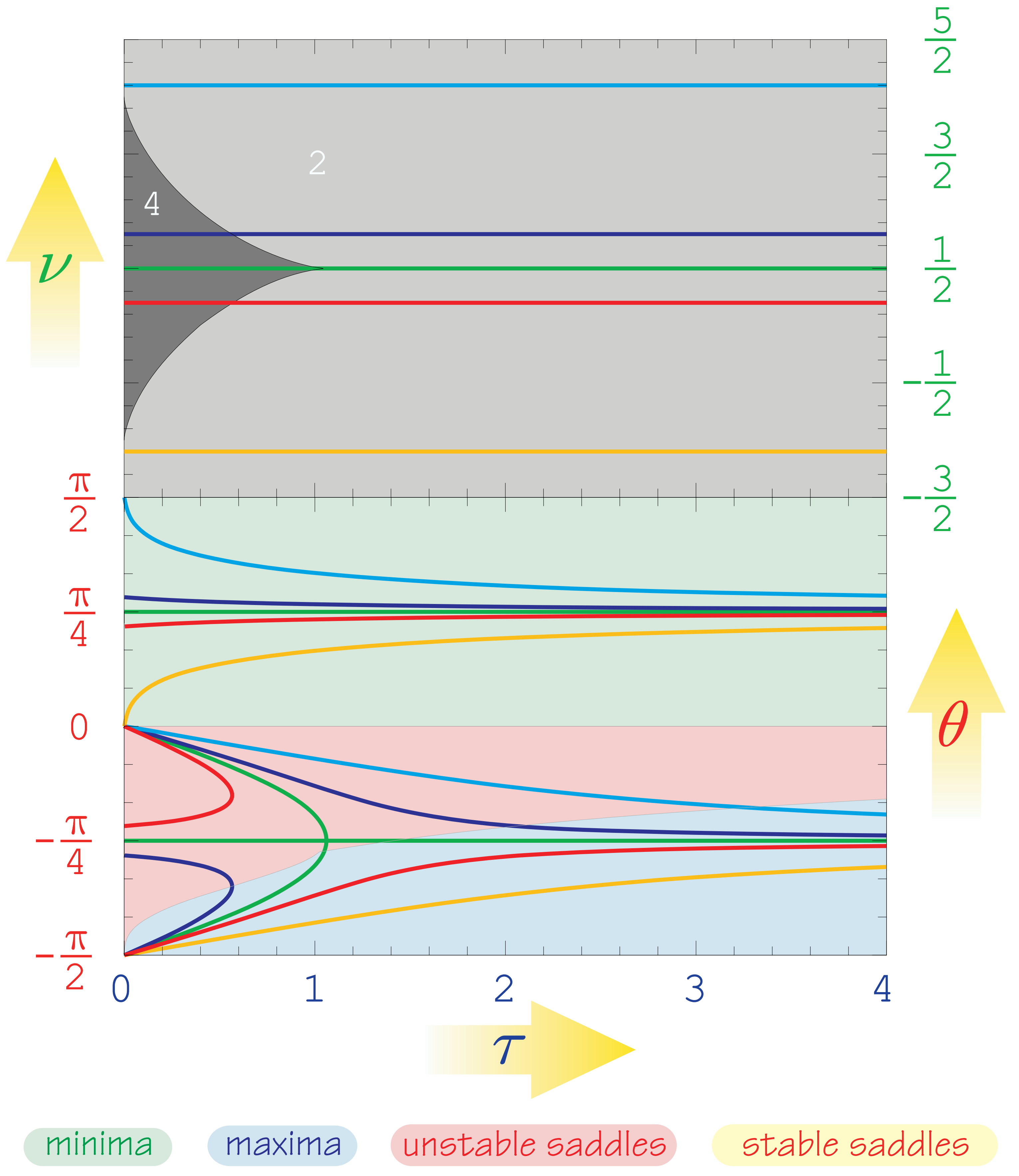
$$-QP \longrightarrow \{\lambda_k\} = \{s_j^2\} \longrightarrow \{s_j\}$$

$$\left.\begin{matrix} \text{max} \\ \text{min} \end{matrix}\right\} \Rightarrow \lambda_j = s_j^2 < 0\;\forall j \Rightarrow \mathrm{Re}(s_j) = 0\;\forall j$$

Always stable

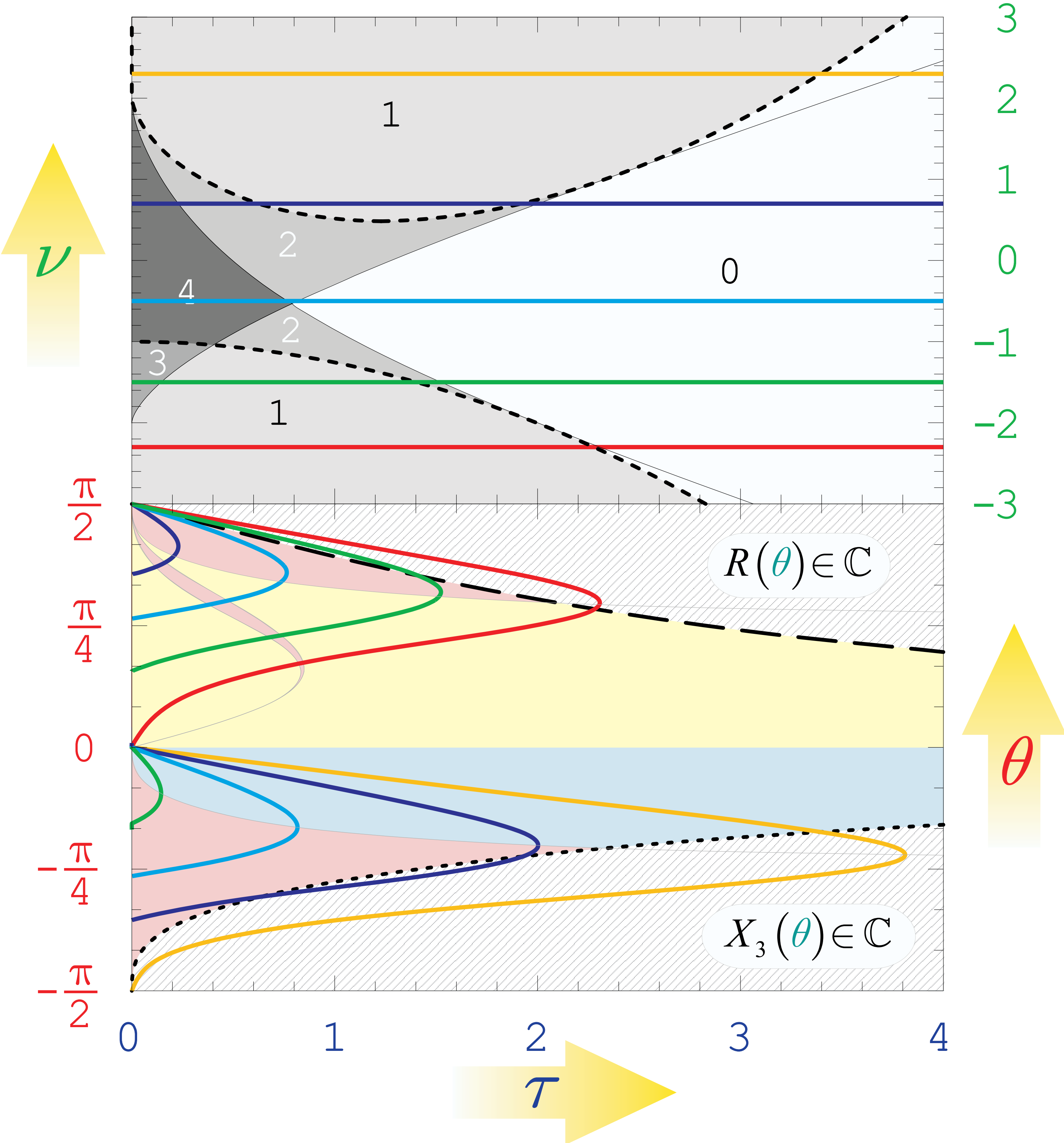
DIMERIC REGIME

$$x_2 \neq 0, \;\; x_1 = x_3 \neq 0$$



NON-DIMERIC REGIME

$x_j \neq 0, \; x_1 \neq x_3$



minima | maxima | unstable saddles | stable saddles

SIMULATIONS: REDUCED DYNAMICS

$$\mathcal{H}\left[\{\zeta_j\}_{j=1}^{3}\right]$$

6 real variables

symplectic reduction

$$\mathcal{H}_{\mathcal{R}}\left[(\xi_1, \phi_1, \xi_2, \phi_2)\right]$$

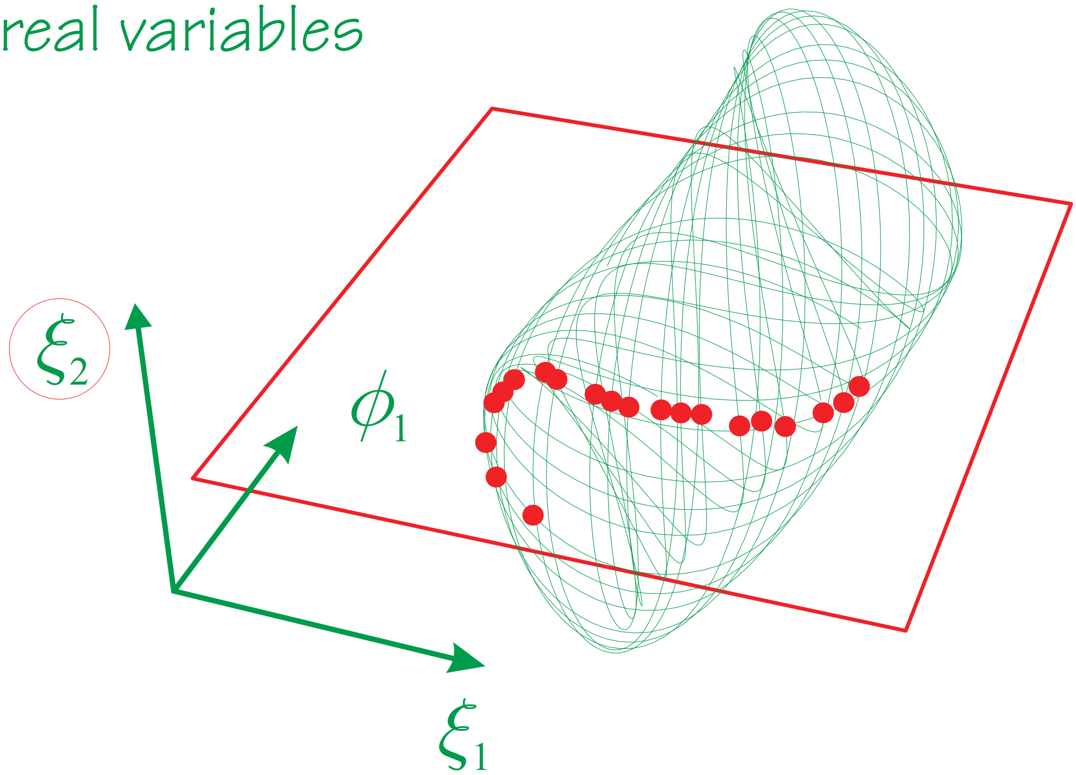
4 real variables

$$\phi_1 = \vartheta_2 - \vartheta_1 \qquad \xi_1 = 1 - 2|\zeta_1|^2$$

$$\phi_2 = \vartheta_3 - \vartheta_2 \qquad \xi_2 = 2|\zeta_3|^2 - 1$$

$$\psi = \tfrac{1}{2}(\vartheta_1 + \vartheta_3) \qquad \rho = |\zeta_1|^2 + |\zeta_2|^2 + |\zeta_3|^2$$

canonically conjugated



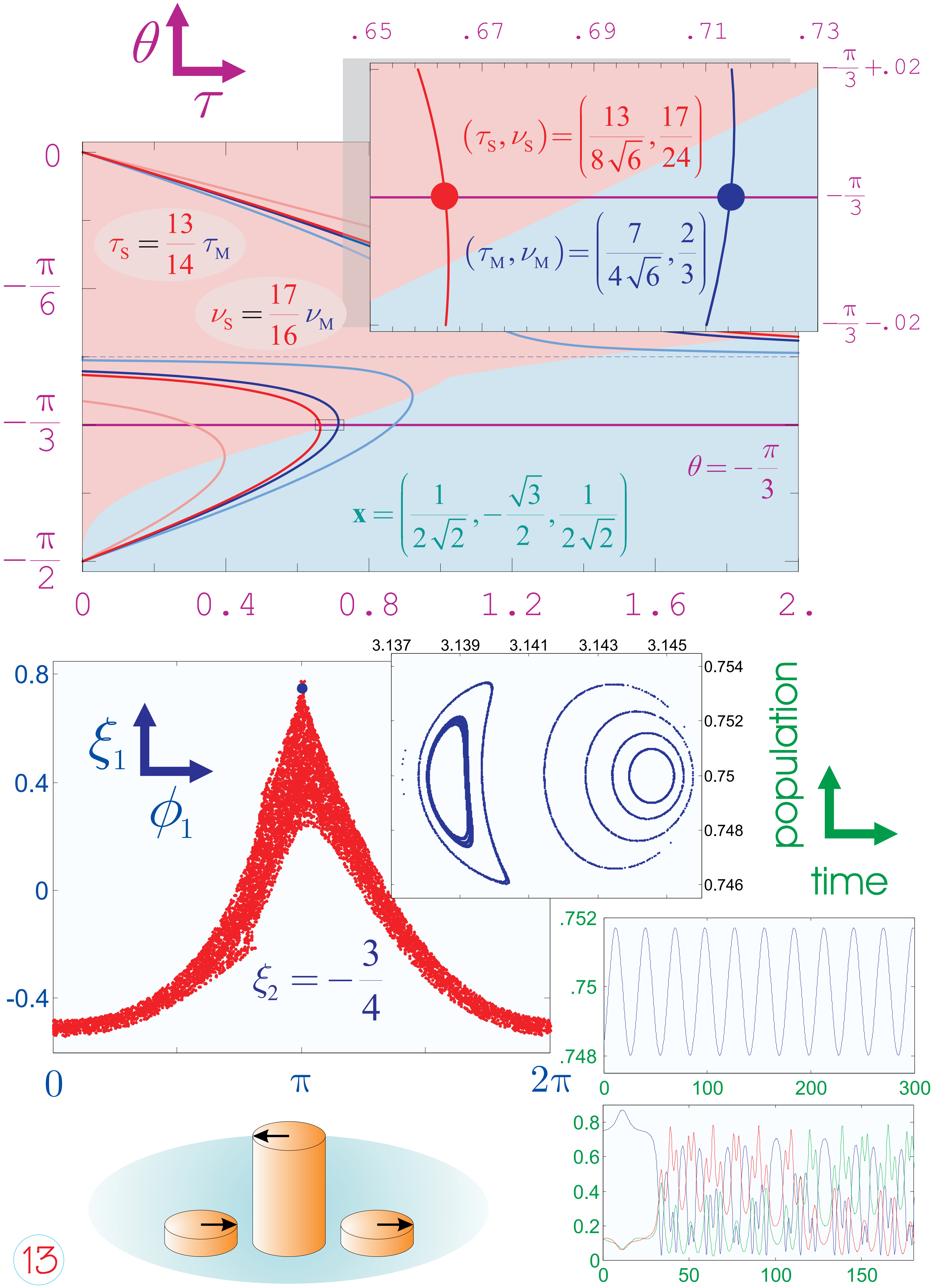
four significant variables (ξ_1, ξ_2, ϕ_1, ϕ_2) conserved (reduced) energy $\mathcal{H}_{\mathcal{R}}$.

Poincare' section

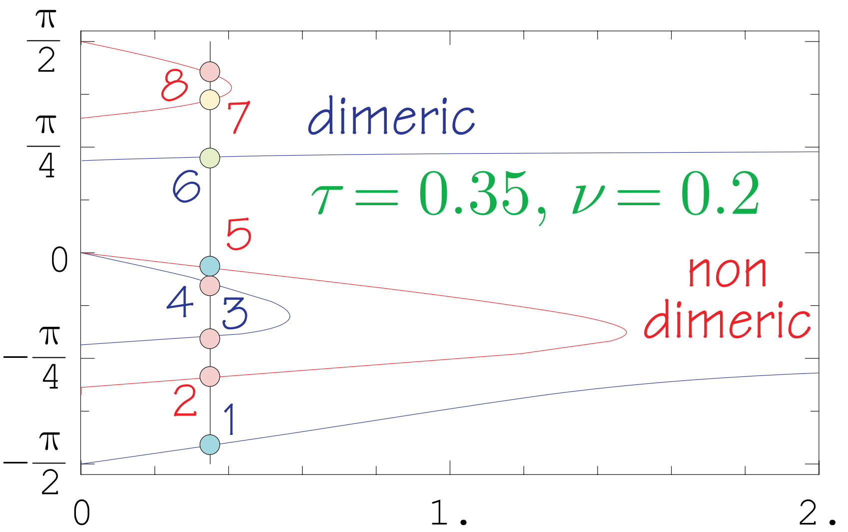
ϕ_1-ξ_1 plane

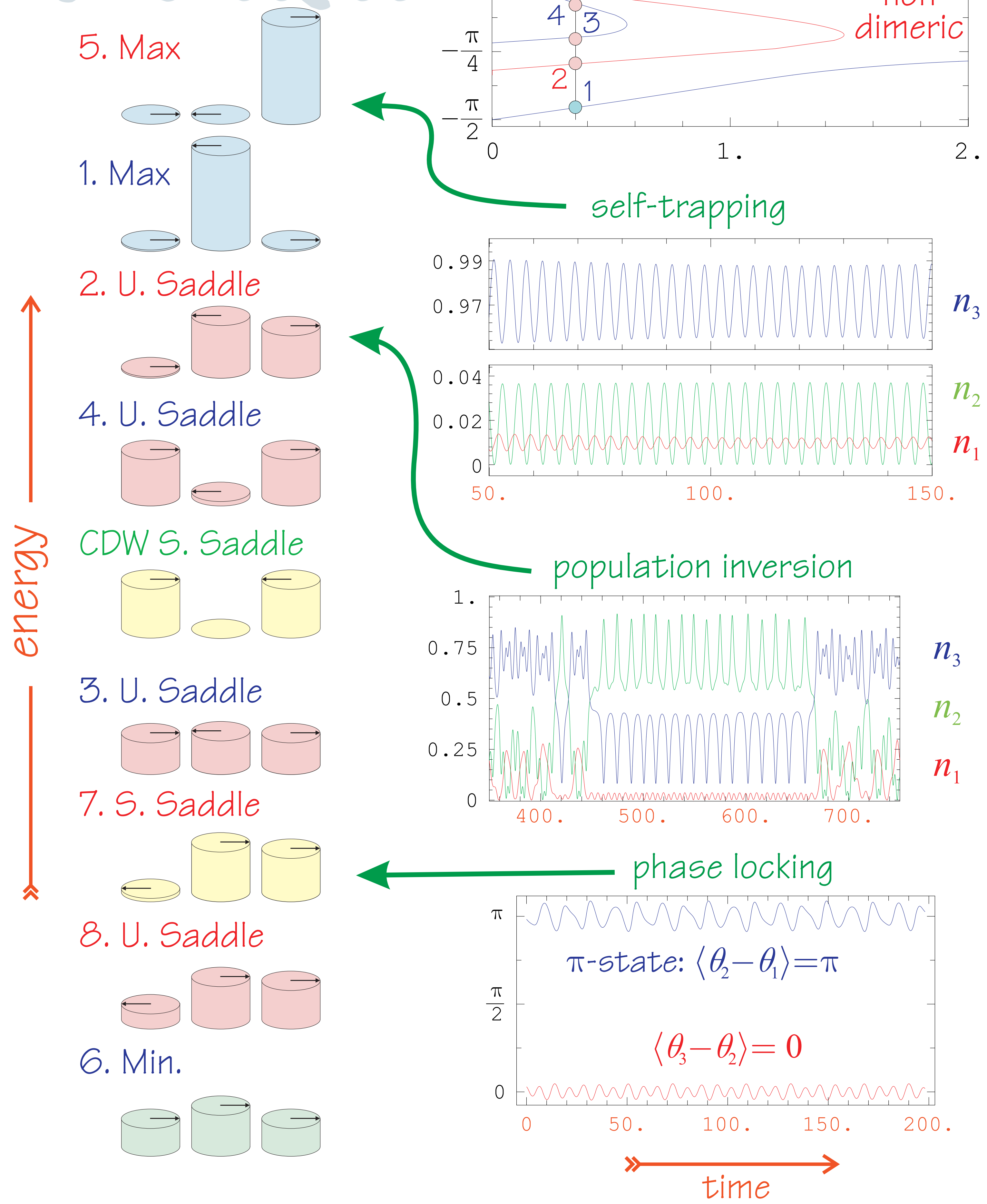
ξ_2 fixed

ϕ_2 dependent


θ
τ
.65 .67 .69 .71 .73
−π/3 +.02
−π/3
−π/3 −.02
(τ_S, ν_S) = (13/(8√6), 17/24)
(τ_M, ν_M) = (7/(4√6), 2/3)
τ_S = 13/14 τ_M
ν_S = 17/16 ν_M
0
−π/6
−π/3
−π/2
θ = −π/3
x = (1/(2√2), −√3/2, 1/(2√2))
0 0.4 0.8 1.2 1.6 2.
3.137 3.139 3.141 3.143 3.145
0.754
0.752
0.75
0.748
0.746
0.8
0.4
0
-0.4
ξ1
φ1
ξ2 = −3/4
0 π 2π
population
time
.752
.75
.748
0 100 200 300
0.8
0.6
0.4
0.2
0
0 50 100 150

MACROSCOPIC EFFECTS





CONCLUSIONS & FUTURE WORK

- Stability diagrams
  - Tool for analyzing and planning experiment.
  - Number, configuration and stability character of the fixed points of the trimeric dynamics for any choice of the parameters (τ, ν).
- Macroscopic effects

  chaos onset, self trapping, population inversion, phase locking

- Identification of further interesting experimental configurations.
- Analysis of the phase dynamics (phase-interference experiments)
- Beyond the mean field
  - Purely quantum analysis
  - Comparison with the mean-field results (mesoscopic crossover)