Field-Theory Approach to Critical Phenomena with Complex Symmetries

Pasquale Calabrese, SNS (Pisa)

Overview of field-theory results for physically interesting Landau-Ginzburg Hamiltonians with complex symmetries, obtained by computing and analyzing high-order perturbative series in the framework of the fixed-dimension and ϵ expansions.

References:

PC, A. Pelissetto & E. Vicari, PRE **65** (2002) 046115.

PC & A. Celi, PRB to appear [cond-mat/0111118].

PC, A. Pelissetto & E. Vicari, cond-mat/0202292.

PC, A. Pelissetto & E. Vicari, cond-mat/0203533.

PC, A. Pelissetto & E. Vicari, Acta Phys. Slov. **52**, 311 (2002) [cond-mat/0205251].

PC, A. Pelissetto & E. Vicari, cond-mat/0209580.

General Review:

A. Pelissetto & E. Vicari, Phys. Rep. **368**, 549 (2002).

Landau-Ginzburg Hamiltonians

In the framework of RG, general Landau-Ginzburg Hamiltonian for a N-component field φ_i

$$\mathcal{H} = \int d^d x \Big\{ \frac{1}{2} \sum_i [(\partial_\mu \varphi_i)^2 + r_i \varphi_i^2] + \frac{1}{4!} \sum_{ijkl} u_{ijkl}\, \varphi_i \varphi_j \varphi_k \varphi_l \Big\}$$

r_i and u_{ijkl} depend essentially on the symmetry.

LGW $\mathcal{H}$ with a single quadratic invariant

In these cases criticality is driven by one parameter, such as T, and all φ_i become critical,

$$r_i = r, \qquad\qquad u_{iikl} \propto \delta_{kl}$$

so that $\sum_i \varphi_i^2$ is the only quadratic invariant

[Brézin, Le Guillou, Zinn-Justin, 1974]

- The simplest case is the $O(N)$ Hamiltonian

$$\mathcal{H} = \int d^d x \left[\frac{1}{2} (\partial_\mu \varphi)^2 + \frac{1}{2} r \varphi^2 + \frac{1}{4!} u \, (\varphi^2)^2 \right]$$

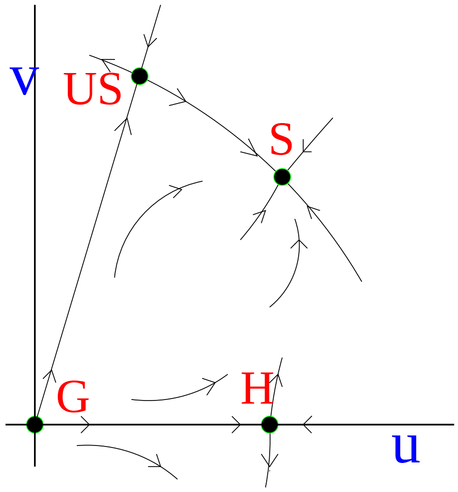
describing the helium superfluid transition, liquid-vapor transitions in fluids, critical behavior of isotropic magnetic materials, etc...

- With increasing N and in the absence of a sufficiently large symmetry, several quartic couplings must be considered.

• These more complex cases are relevant for many magnetic and structural transitions.

• In the case of a continuous transition, the critical behavior is described by the stable FP.

Example with two couplings



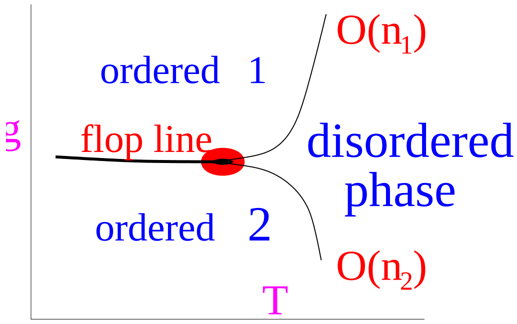
• In the absence of a stable FP, a first-order transition is expected, even in the case mean field suggests a continuous transition (first-order transition induced by fluctuations).

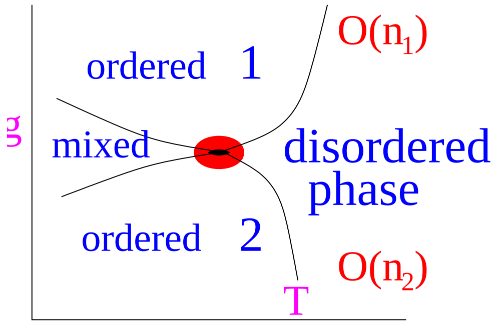
• Even in the presence of a stable FP, systems that are outside its attraction domain are expected to undergo a first order transition.

• More than four FP may exist.

LGW $\mathcal{H}$ with more quadratic invariants

More complex Hamiltonians are required to describe the critical behavior at a multicritical point arising from the competition of distinct types of ordering, when two critical lines meet in the phase diagram.





$$\mathcal{H}(x) = \frac{1}{2}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + r_1\phi_1^2 + r_2\phi_2^2\right] + \\ +V_1(\phi_1) + V_2(\phi_2) + W(\phi_1,\phi_2)$$

where ϕ_1, ϕ_2 are n_1-,n_2-component fields.

- A multicritical point is approached by tuning two relevant parameters, related to r_1 and r_2.

e.g. Anisotropic Ferromagnets

- The multicritical behavior is determined by the stable FP of the theory when both r_1 and r_2 are tuned to their critical value.

Field-Theory Perturbative Expansions

The RG flow, the critical exponents corresponding to each FP, etc..., can be determined using field-theory perturbative methods.

• ϵ expansion [Wilson, Fisher, 1972]

the minimal subtraction scheme is particularly convenient, especially for high-order calculations.

• Fixed-dimension expansion in powers of the zero-momentum quartic couplings g_{ijkl} [Parisi, 1973]

based on zero-momentum renormalization conditions in the HT phase, relating m, g_{ijkl} to the Hamiltonian parameters r, u_{ijkl}

$$\Gamma^{(2)}_{ij}(p) = \delta_{ij} Z_\phi^{-1} \left[m^2 + p^2 + O(p^4) \right]$$

$$\Gamma^{(4)}_{ijkl}(0) = m\, Z_\phi^{-2}\, g_{ijkl} \qquad \beta_{ijkl}(g_{abcd}) = m \left. \frac{\partial g_{ijkl}}{\partial m} \right|_{u_{abcd}}$$

FP's are obtained from the zeros g^*_{ijkl} of β_{ijkl}.

• FT expansions are asymptotic $\Rightarrow$ resummation of the series exploiting Borel summability, and the knowledge of the large-order behavior determined by steepest-descent calculations.

For $O(N)$ models, accurate results from series up to

$O(\epsilon^5)$ in ϵ exp [Chetyrkin et al,1983; Kleinert et al,1991]

$O(g^6)$ in $d = 3$ exp [Baker Jr, et al, 1977]

$O(g^5)$ in $d = 2$ exp [Orlov, Sokolov, 2000]

We have performed high-order computations for a number of LGW Hamiltonians, typically

$\longrightarrow$ to five loops in the ϵ exp (≈ 200 diagrams)
$\longrightarrow$ to six loops in the $d = 3$ exp (≈ 1000 diagrams)
$\longrightarrow$ to five loops in the $d = 2$ exp (≈ 200 diagrams)

Beside improving the accuracy, in some cases high-order calculations are needed to determine the physical picture in $d = 3$.

- LGW $\mathcal{H}$ with cubic anisotropy

$$\mathcal{H}(x) = \frac{1}{2}\left[(\partial_\mu\phi)^2 + r\phi^2\right] + u(\sum_i \phi_i^2)^2 + v\sum_i \phi_i^4$$

$\longrightarrow$ Magnets with cubic anisotropy, . . .

- MN model $\phi_{a,i}$ with $a = 1, ...M$, $i = 1, ...N$

$$\mathcal{H}_{MN}(x) = \frac{1}{2}\sum_{i,a}\left[(\partial_\mu\phi_{a,i})^2 + r\phi_{a,i}^2\right] + \sum_{ij,ab}(u + v\delta_{ij})\,\phi_{a,i}^2\phi_{b,j}^2$$

$\longrightarrow$ For $N \to 0$, randomly diluted M-component spin systems, quenched disorder coupled to the energy density.

-

$$\mathcal{H} = \sum_{i,a}\frac{1}{2}\left[(\partial_\mu\phi_{a,i})^2 + r\phi_{a,i}^2\right] + \sum_{ij,ab}(u + v\delta_{ij} + w\delta_{ij}\delta_{ab})\,\phi_{a,i}^2\phi_{b,j}^2$$

$\longrightarrow$ For $N \to 0$, randomly dilute M-component cubic model.
$\longrightarrow$ For $N = 2$, structural and magnetic transitions.

- $O(M) \otimes O(N)$ LGW $\mathcal{H}$ for $\phi_{a,i}$ with $a = 1, ...M$, $i = 1, ...N$

$$\frac{1}{2}\sum_{a,i}[(\partial_\mu\phi_{a,i})^2 + r\phi_{a,i}^2] + u(\sum_{a,i}\phi_{a,i}^2)^2 + v\sum_{a,b}(\sum_i \phi_{a,i}\phi_{b,i})^2$$

$\longrightarrow$ For $M = 2$, $O(N) \to O(N-2)$ symmetry breaking, N-component frustrated systems with noncollinear order.

[See Parruccini talk]

- $O(n_1) \oplus O(n_2)$ LGW $\mathcal{H}$ for ϕ_1, ϕ_2 with n_1-, n_2-components

$$\frac{1}{2}[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + r_1\phi_1^2 + r_2\phi_2^2] + u_1(\phi_1^2)^2 + u_2(\phi_2^2)^2 + 2w\phi_1^2\phi_2^2$$

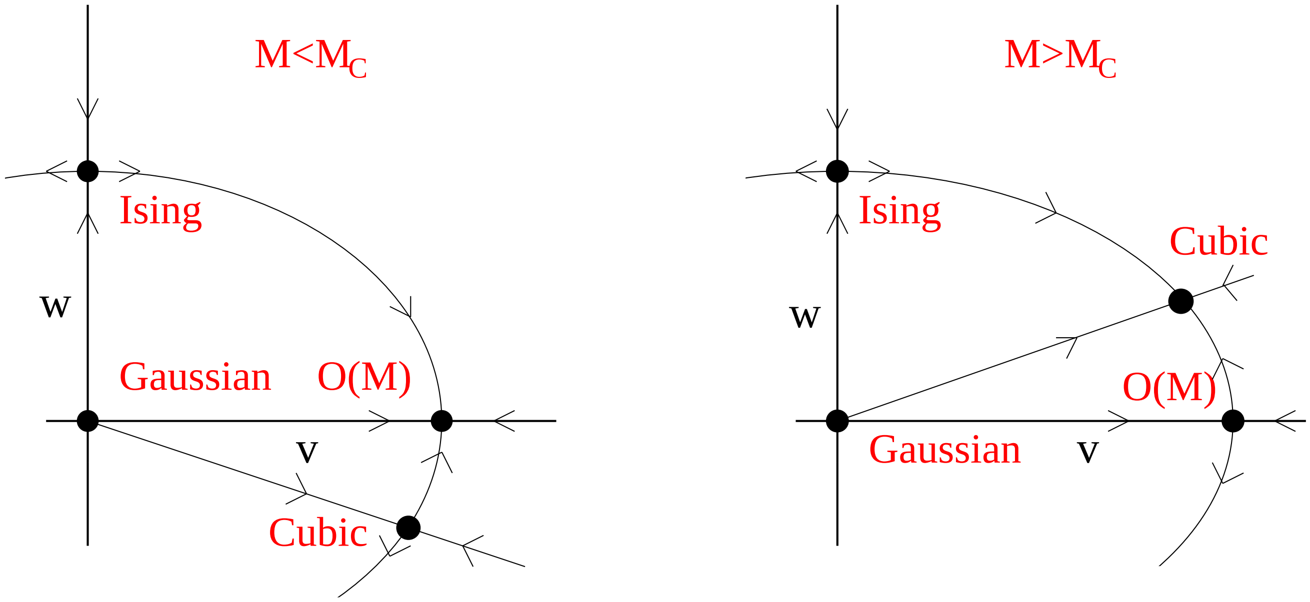
$\longrightarrow$ Multicritical phenomena arising from the competition of distinct types of ordering, with n-comp order parameters and $O(n)$ symmetries, when two critical lines meet in the phase diagram

Cubic Models

$$\mathcal{H}_c = \frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{1}{2} r \phi(x)^2 + \frac{1}{4!} v_0 \left[\phi(x)^2\right]^2 + \frac{1}{4!} w_0 \sum_{i=1}^{M} \phi_i(x)^4$$

$\longrightarrow$ Relevant for magnets with cubic anisotropy, arising from the lattice structure or single ion terms

$\longrightarrow$ Three- and four-state AF Potts models for $N = 2, 3$



Higher loop calculation leads to $\boxed{M_c = 2.89(4)}$

[Carmona, Pelissetto & Vicari 2000]

In the physical relevant case of $M = 3$ the cubic fixed point is very close to the $O(3)$ one

$\Rightarrow$ Exponents are practically indistinguishable

Precise estimates of differences between exponents are obtained expanding around the $O(3)$ fixed point:

$$\nu_c - \nu_H = -0.0003(3)$$
$$\eta_c - \eta_H = -0.0001(1)$$
$$\gamma_c - \gamma_H = -0.0005(7)$$

[PC, Pelissetto, and Vicari 2002]

In 2D for $N = 2$ a line of fixed points exists.

[PC & Celi 2001]

Randomly dilute spin systems

$$\mathcal{H}_p = J \sum_{<ij>} \rho_i \, \rho_j \; \vec{s}_i \cdot \vec{s}_j,$$

$\longrightarrow$ Mixing of antiferromagnetic materials with non magnetic ones, such as $Fe_pZn_{1-p}F_2$, $Mn_pZn_{1-p}F_2$.

Replica trick to handle the quenched disorder

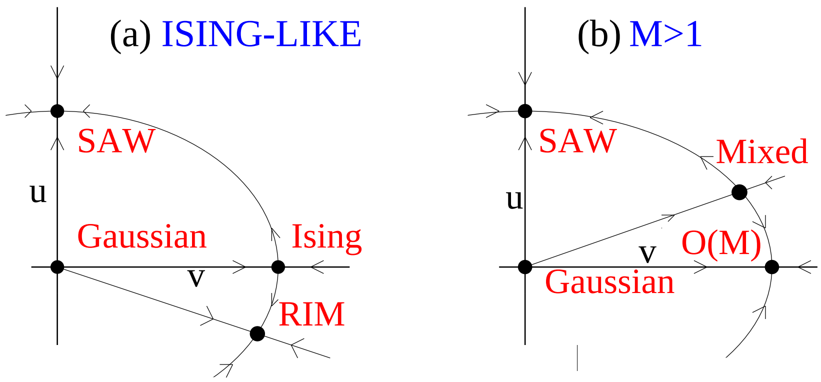
$$\mathcal{H}_{MN} = \frac{1}{2} \sum_{i,a} \left[(\partial_\mu \phi_{a,i})^2 + r \phi_{a,i}^2 \right] + \sum_{ij,ab} (u + v\delta_{ij}) \, \phi_{a,i}^2 \phi_{b,j}^2$$

where $a, b = 1, ...M$, $i, j = 1, ...N$, $u < 0$. The original system is recovered in the limit $N \to 0$.

Harris Criterion $\Rightarrow$ the pure fixed point is stable if $\alpha < 0$.

RG flow for $N \to 0$ from FT

confirming Harris criterion



Stability of the pure $O(M)$ FP for $M > 1$.

RIM universality class for Ising-like systems along $T_c(p)$.

	γ	ν	β
six-loop $d = 3$ FT	1.330(17)	0.678(10)	0.349(5)
experiments	1.34(6)	0.70(2)	0.36(2)
Monte Carlo	1.342(10)	0.684(5)	

exper. $\to$ [Slanič et al, 1999], MC $\to$ [Ballesteros et al, 1998]

Quenched Randomness and Cubic Anisotropy

$$\mathcal{H}_e = \sum_{i,a} \frac{1}{2} \left[(\partial_\mu \phi_{a,i})^2 + r\phi_{a,i}^2 \right] + \sum_{ij,ab} (u_0 + v_0 \delta_{ij} + w_0 \delta_{ij}\delta_{ab})\, \phi_{a,i}^2 \phi_{b,j}^2$$

3D $O(M)$ and cubic models with $M \geq 2$ have $\alpha < 0$.

This is not the end of the story.

Randomness may change the attraction domain of this or/and may give rise to new fixed points: softening of the first-order transition of the pure systems that are outside the attraction domain of the stable fixed point is possible.

In 2D it is rigorously proved that all the first order transitions are softened by quenched randomness coupled to the local energy density.

[Aizenman & Wehr, Hui & Berker '89]

For the dilute cubic models (with $v_0 < 0$) it was argued that in $(2+\epsilon)$D the transition is softened whereas in $4-\epsilon$ no.

[Cardy '96]

Analyzing six-loop three-dimensional series we conclude that in 3D there is no softening for dilute cubic models.

[PC, Pelissetto & Vicari '02]

Stability of the O(N)-symmetric theory with respect to general perturbations

Consider a perturbation $P(x)$ at the $O(N)$ FP

$$\mathcal{H} = \mathcal{H}_{O(N)} + h_p \sum_x P(x), \qquad \mathcal{F}_{\rm sing} \approx t^{d\nu} f(h_p t^{-\phi_o})$$

If $\phi_p > 0$ then P is relevant, if $\phi_p < 0$ it is irrelevant.

It is convenient to distinguish the perturbations using the power m of the field and the spin l $(m > l)$ of the representation of the $O(N)$ group, because for a given m, ϕ_o depends only on the spin value

$$\begin{array}{ll} \text{spin } 2: & P^{ab}_{2,2} = \phi^a\phi^b - \frac{1}{N}\delta^{ab}\phi^2. \\ \text{spin } 4: & P^{abcd}_{4,4} = \phi^a\phi^b\phi^c\phi^d - ... \\ \vdots & \vdots \end{array}$$

$P^{ab}_{2,2}$ is relevant for the description of the breaking of the $O(N)$ symmetry down to $O(M) \oplus O(N-M)$, $N > M$.

$\longrightarrow$ The cubic term transform as a spin-4 representation $\Rightarrow$ For $N \geq 3$ the $O(N)$ theory is unstable with respect to quartic spin-4 perturbations.

N		ϕ_2	ϕ_4
2	six-loop $d=3$ exp	1.184(12)	$-0.069(5)$
	$O(\epsilon^5)$	1.174(12)	$-0.076(3)$
3	six-loop $d=3$ exp	1.271(21)	0.009(4)
	$O(\epsilon^5)$	1.26(2)	0.002(3)
5	six-loop $d=3$ exp	1.40(4)	0.145(7)
	$O(\epsilon^5)$	1.40(3)	0.15(1)

[PC, Pelissetto & Vicari 2002]

Multicritical phenomena

Multicritical point (MCP) arising from the competition of two distinct order parameters, where two critical lines characterized by $O(n_1)$ and $O(n_2)$ symmetry meet.

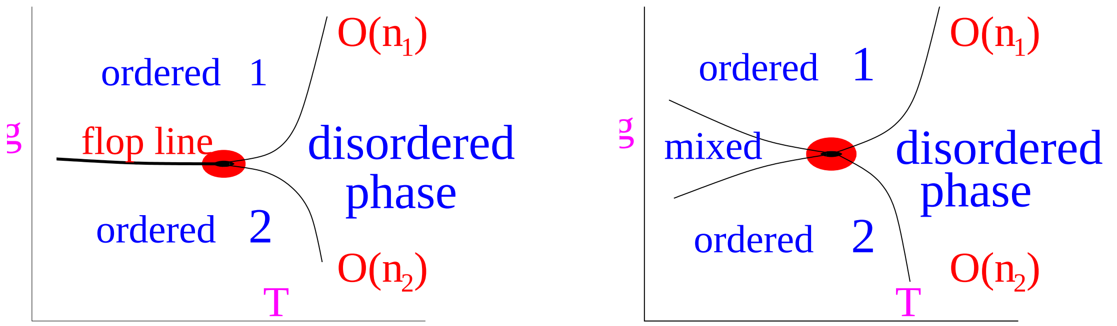
$O(n_1)\oplus O(n_2)$ LGW $\mathcal{H}$ [Kosterlitz, Nelson & Fisher '74]

$$\mathcal{H} = \frac{1}{2}[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + r_1\phi_1^2 + r_2\phi_2^2] + u_1(\phi_1^2)^2 + u_2(\phi_2^2)^2 + 2w\phi_1^2\phi_2^2$$

ϕ_1, ϕ_2 are n_1-, n_2-comp fields

Mean-field analysis

Defined $\Delta = u_1 u_2 - w^2$



$\Delta \le 0$ $\Delta > 0$

[Liu, Fisher, 1972]

The critical behavior at the MCP is determined by the RG flow of the theory when both r_1 and r_2 are tuned to their critical value.

An interesting possibility is that at the MCP

$$O(n_1)\oplus O(n_2) \longrightarrow O(N) \qquad N = n_1 + n_2$$

requiring the stability of the $O(N)$ FP. According to mean field, it gives rise to a bicritical point, with a first-order flop line separating the two ordered phases.

Low-order calculations $\Rightarrow$ three non-trivial FP:

- "bicritical" $O(N)$ FP, stable for $N < N_c = 4 - 2\epsilon$ (suggesting that $N_c > 3$ in $d = 3$).
- With increasing N, the stability passes to the biconal FP (tetracritical, extending mean-field arguments)
- Finally ($n_1 n_2 + 2(n_1 + n_2) > 32 + O(\epsilon)$) to the tetracritical decoupled FP where each order parameter maintains its own critical behavior.

[Kosterlitz, Nelson & Fisher '74]

$\boxed{\text{High-order analyses contradict these predictions.}}$

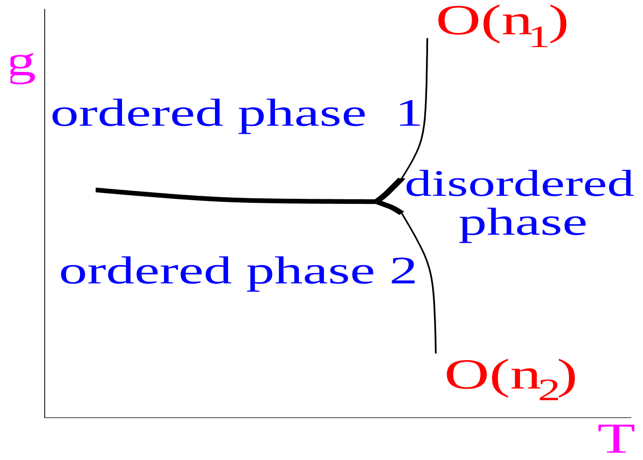
$\longrightarrow$ Our six-loop fixed-d and $O(\epsilon^5)$ computations show that $N_c \sim 2.9$ for $d = 3$ (the enlargement of the symmetry occurs only when two Ising lines meet).

$\longrightarrow$ For $N \geq 4$ nonperturbative arguments show that the DFP is stable. [Aharony '02]

In the $N \geq 3$ cases, the $O(N)$ critical behavior at the MCP needs a further tuning to be realized. The crossover exponent related to this further RG instability (spin-4 quartic perturbation) is small

e.g. $\phi = 0.009(4),\ 0.145(7)$ for $N = 3, 5$.

When the initial parameters of the Hamiltonian are not in the attraction domain of the stable FP, the transition between the disordered and ordered phases should be first order in the neighborhood of the MCP. A possible phase diagram is



where tricritical points appear along the critical lines.

Applications

$\longrightarrow$ $\boxed{1 \oplus 2}$ Anisotropic antiferromagnets in a uniform magnetic field along the anisotropy axis in the $T - H_{\|}$ phase diagram.

The MCP occurs when a XY and an Ising critical line meet.

Our $O(\epsilon^5)$ calculations show that the stable FP is the tetracritical (according to mean-field) Biconal FP.

Hardly distinguishable from the Heisenberg critical behavior: the critical exponents very close to the Heisenberg ones, e.g. $|\nu_{\rm bi} - \nu_H| < 0.001$. The crossover exponent at the $O(3)$ FP associated with the additional RG instability is very small: $\phi = 0.009(4)$

$\longrightarrow$ $\boxed{2 \oplus 2}$ Multicritical behavior is expected at the nematic-smetic-A-smetic-C transition in liquid crystals and more generally in the presence of ferromagnetic and nematic interactions.

Our results show that the MFP governing these transitions is the Decoupled one (with very strong crossover effects with both the other FP).

$\longrightarrow$ $\boxed{1 \oplus 1}$ Qualitatively agreement with one-loop result (stability of XY-like FP).

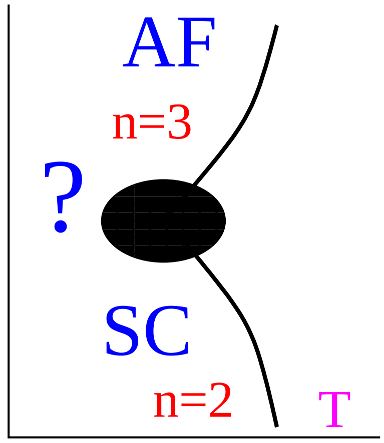
$\longrightarrow$ $\boxed{0 \oplus 0}$ Copolymer networks (work in progress)

$\longrightarrow$ High-T$_c$ superconductivity

The $SO(5)$ theory provides a unified description of the 3-component antiferromagnetic and the complex superconductor order parameters, combining them into a 5-component theory.

[Zhang '97]

Within some assumptions, the phase diagram in the T-doping plane has been discussed using the $O(3) \oplus O(2)$ LGW $\mathcal{H}$. It has been argued, and apparently supported by recent Monte Carlo simulations [Hu '01], that the $O(5)$ symmetry is realized at a bicritical point where the AF and SC critical lines meet.



Nonperturbative arguments show the tetracritical decoupled FP is stable.

[Aharony '76 & '02]

Our FT analysis shows that the bicritical $O(5)$ FP is unstable, and the $O(5)$ symmetry at the MCP can be realized only by tuning an additional parameter.

The Biconal FP is unstable ($\omega_i = -0.07(5)$).

The multicritical behavior is described by this FP or it is of first-order type. In the case of soft breaking of $O(5)$ at the MCP the $O(5)$ FP can be still describe the behavior the preasymptotic region, due to the fact that the crossover exponent associated to the additional instability is $\phi = 0.145(7)$. The extension of the preasymptotic region depends on non-universal parameter.