Numerical studies of supersymmetric Yang-Mills quantum mechanics

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[J. W., M. C., 2002, 2004]

Supersymmetric Yang-Mills Quantum Mechanics (SYMQM) is obtained from dimensional reduction of a supersymmetric Yang-Mills theory from $d+1$ to $0+1$ dimensions.

Much interest into SYMQM stems from the BFSS conjecture [T. Banks, W. Fischler, S. Shenker, L. Susskind, 1997]:

> The $N \to \infty$ limit of $\mathrm{U}(N)$ $d = 9$ SYMQM
> is equivalent to M-theory (in flat space-time).

Susskind proposed a further conjecture [L. Susskind, 1997]:

> $\mathrm{U}(N)$ $d = 9$ SYMQM at finite N is equivalent
> to a well-defined sector of M-theory.

In the $A_0 = 0$ gauge, we obtain for SYMQM the Hamiltonian

$$H = \mathrm{tr}\big\{ P_i P_i - \tfrac{1}{2} g^2 [X_i, X_j][X_i, X_j] + g \Theta^T \Gamma_i [\Theta, X_i] \big\},$$

where, in matrix notation,

$$P_i = p_a^i T_a, \quad X_i = x_a^i T_a, \quad \Theta^\alpha = \theta_a^\alpha T_a,$$

$i = 1, \ldots, d$, Γ_i are the d+1-dimensional Dirac α matrices, and θ_a^α are d+1-dimensional Majorana (Hermitian) spinors. All dynamical variables obey canonical (anti)commutation rules

$$[x_a^i, p_b^j] = i\delta^{ij}\delta_{ab}, \quad \{\theta_a^\alpha, \theta_b^\beta\} = \delta^{\alpha\beta}\delta_{ab}.$$

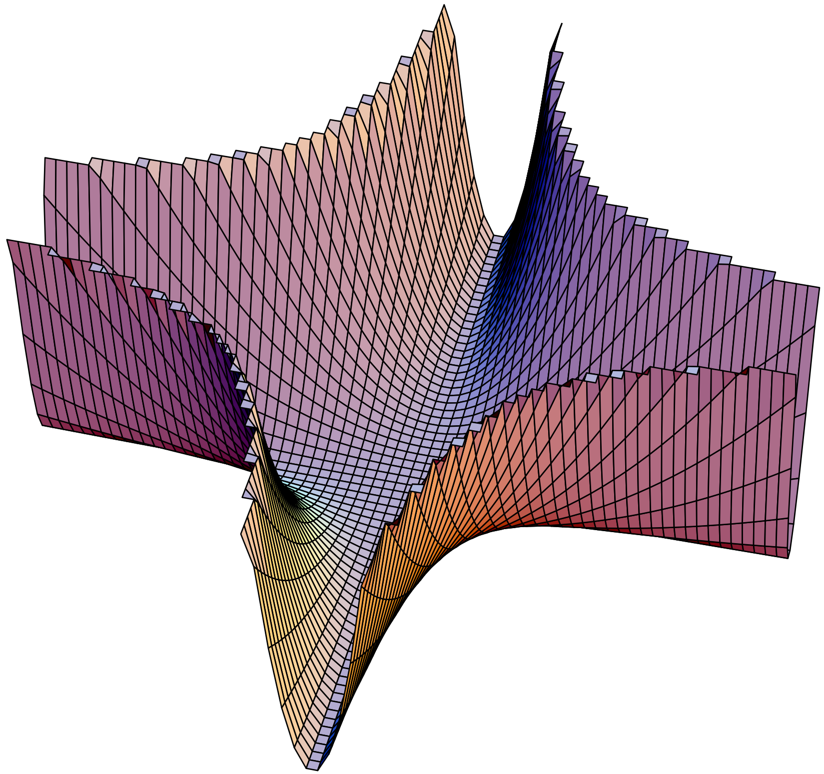
The rotational symmetry of the original theory becomes an internal $\mathrm{O}(d)$ symmetry.

The gauge symmetry becomes a "rigid" symmetry of SYMQM; the condition of gauge invariance becomes a constraint limiting the physical space of SYMQM to the gauge-invariant subspace (i.e., the subspace of gauge singlets).

SYMQM is supersymmetric (in the gauge-invariant sector) for $d = 1$, 2, 3, 5 and 9.

Maldacena introduced a $d = 9$ model with additional terms in the Hamiltonian, breaking the $\mathrm{O}(9)$ symmetry down to $\mathrm{O}(3)\times\mathrm{O}(6)$, to describe in the $N \to \infty$ limit M-theory in the background of a supersymmetric plain wave [D. Berenstein, J.M. Maldacena, H. Nastase, 2002].

The potential $-\frac{1}{2}g^2\,\mathrm{tr}\{[X_i, X_j][X_i, X_j]\}$ is quartic, non-negative, and grows like r^4 for large r, excluding “valleys” with $[X_i, X_j] = 0$, where the potential behaves like $r^2 x_\perp^2$. The model obtained eliminating fermionic variables has a discrete spectrum, since the valleys get narrower with increasing r, originating a zero-point energy – and therefore an effective potential – growing linearly with r. In the supersymmetric model the zero-point energy vanishes, the is no confining potential, and the model has a continuum spectrum (in addition to discrete levels).



From the viewpoint of M-theory, one of the most relevant questions about SYMQM is the existence of zero-energy threshold bound states, which can be identified with the supergraviton.

Several arguments point to the existence of threshold bound states only for $d = 9$ [M.B. Halpern, C. Schwartz, 1998; S. Sethi, M. Stern, 1998].

We are interested not only in existence: if SYMQM is relevant for M-theory, and if M-theory is relevant for nature, the spectrum, detailed shape of the states, etc... are very important.

We developed a numerical approach to the study of SYMQM: rewrite the bosonic variables x_b^i and p_b^i in terms of creation and annihilation operators

$$x_b^i = \frac{1}{\sqrt{2}}(a_b^i + a_b^{i\dagger}), \quad p_b^i = \frac{1}{i\sqrt{2}}(a_b^i - a_b^{i\dagger}); \quad [a_b^i, a_c^{k\dagger}] = \delta_{bc}^{ik}$$

and truncate the Hilbert space to a maximum number of bosonic quanta:

$$n_B \equiv a_b^{i\dagger} a_b^i \leq n_{B\,\max};$$

compute the matrix elements of H (and any other operator) in the occupation number basis and diagonalize H numerically.

We expect for the energy levels of the truncated Hilbert space $E_n(n_{B\,\max})$

$$E_n(n_{B\,\max}) \sim E_n(\infty) + c\exp(-bn_{B\,\max}) \quad \text{(discrete spectrum)},$$

$$E_n(n_{B\,\max}) \sim \frac{1}{n_{B\,\max}}\,\rho\left(\frac{n^\nu}{n_{B\,\max}}\right) \quad \text{(continuum spectrum)}.$$

The truncation preserves gauge symmetry and rotational symmetry, but not supersymmetry, which is recovered in the $n_{B\,\max} \to \infty$ limit.

In the case of an $\mathrm{SU}(2)$ gauge group, we write

$$\begin{aligned} H &= H_K + H_P + H_F, \\ H_K &= \tfrac{1}{2} p_a^i p_a^i, \\ H_P &= \tfrac{1}{4} g^2 \epsilon_{abc} \epsilon_{ade} x_b^i x_c^j x_d^i x_e^j, \\ H_F &= \tfrac{1}{2} i g \epsilon_{abc} \theta_a^T \Gamma^k \theta_b x_c^k; \end{aligned}$$

the generators of gauge transformations are

$$G_a = \epsilon_{abc} \big(x_b^k p_c^k - \tfrac{1}{2} i \theta_b^T \theta_c \big),$$

SUSY generators are

$$Q_\alpha = \Gamma^k \theta_a p_a^k + i g \epsilon_{abc} \Sigma^{jk} \theta_a x_b^j x_c^k,$$

where $\Sigma^{jk} = -\frac{1}{4} i [\Gamma^j, \Gamma^k]$, and SUSY algebra is

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta} H + 2g \Gamma^k_{\alpha\beta} x_a^k G_a.$$

In order to compute the matrix elements efficiently, it is crucial to avoid completely gauge-variant states and to preserve rotational symmetry at all steps of the computation.

We illustrate the computation for the case $\mathrm{SU}(2)$ in $d = 1$: this model is free ($H_P = 0$), but still nontrivial due to the gauge-invariance constraint.

Replacing the two-component Majorana fermion θ_a with a one-component Dirac fermion ψ_a, we can write

$$H = \tfrac{1}{2} p_a p_a + i g \epsilon_{abc} \psi_a^\dagger x_b \psi_c, \qquad Q = \psi_a p_a;$$

We introduce creation and annihilation operators a, $a^\dagger$ for x and p; the fermionic annihilation operator is simply $f = \psi$, since $\{f_b, f_c^\dagger\} = \delta_{bc}$.

The fermion number $n_F = f_b^\dagger f_b$ is conserved; thanks to the particle-hole symmetry $n_F \to 3 - n_F$, it is sufficient to study the sectors $n_F = 0,\ 1$.

The boson number is $n_B = a_b^\dagger a_b \equiv B - 3$.

We introduce bilinear gauge-invariant creation and annihilation operators

$$A = a_b a_b, \quad A^\dagger = a_b^\dagger a_b^\dagger, \qquad F = a_b f_b, \quad F^\dagger = a_b^\dagger f_b^\dagger, \qquad (f_b^\dagger f_b^\dagger = 0);$$

they satisfy the (anti)commutation rules

$$[A, A^\dagger] = 4B + 6, \qquad [A, B] = 2A, \qquad \{F, F^\dagger\} = 2B.$$

The trilinear gauge-invariant creation operators

$$\epsilon_{abc} a_a^\dagger f_b^\dagger f_c^\dagger, \quad \epsilon_{abc} f_a^\dagger f_b^\dagger f_c^\dagger, \qquad (\epsilon_{abc} a_a^\dagger a_b^\dagger = 0)$$

are only needed to generate the states of the $n_F = 2,\ 3$ sectors (note that $\left(F^\dagger\right)^2 = 0$).

We can generate an orthonormal basis of the space of gauge-invariant states with $n_F = 0,\, 1$ applying $A^\dagger$ and $F^\dagger$ to the vacuum: denoting the states by $|n_F, n_B\rangle$,

$$|2n+m, m\rangle \equiv \frac{1}{\sqrt{c_{2n+m,m}}} \left(A^\dagger\right)^n \left(F^\dagger\right)^m |0,0\rangle.$$

The coefficient can be computed recursively: defining

$$\langle 0| A^n A^\dagger \equiv l_n \langle 0| A^{n-1},$$

clearly $c_{2n,0} = l_n\, c_{2n-2,0}$; exploiting the above commutators and $\langle 0,0|B = 0$, we obtain $l_1 = 6$ and

$$\begin{aligned} l_n \langle 0|A^{n-1} &= \langle 0|A^{n-1} A A^\dagger = \langle 0|A^{n-1}(A^\dagger A + 4B + 6) \\ &= \langle 0|[(l_{n-1} + 6)A^{n-1} + 4([A^{n-1}, B] + BA^{n-1})] \\ &= (l_{n-1} + 6 + 8(n-1))\langle 0|A^{n-1}; \end{aligned}$$

finally, $l_n = 2n + 4n^2$.

In the $n_F = 0$ sector, we can write

$$H = -\tfrac{1}{4}(A + A^\dagger - 2B + 3)$$

and obtain immediately the matrix elements of H:

$$\begin{aligned} \langle 2n, 0 | H | 2n{-}2, 0\rangle &= \langle 2n{-}2, 0 | H | 2n, 0\rangle = -\tfrac{1}{4}\sqrt{2n + 4n^2}, \\ \langle 2n, 0 | H | 2n, 0\rangle &= n + \tfrac{3}{4}. \end{aligned}$$

The computation of the matrix elements of N in the $n_F = 1$ sector and of the matrix elements of Q between the sectors $n_F = 0$ and $n_F = 1$ is very similar.

Since N has a tridiagonal structure, it can be diagonalized using the $O(N^2)$ algorithm implemented in the `lapack` library, obtaining all eigenvalues for $n_{B\,\mathrm{max}} = 10^5$ in a few minutes on a PC.

The regularized Witten index is defined as

$$I_W(t) = \sum_i (-1)^{n_F(i)} e^{-tE(i)}$$

In the case of discrete spectrum, SUSY implies that states with positive energy cancel out in pairs; therefore

$$I_W(t) = \sum_{i:E(i)=0} (-1)^{n_F(i)},$$

independently of t; $I_W(t) \neq 0$ signals unbroken SUSY.

Due to the particle-hole symmetry, $I_W(t)$ vanishes identically for $\mathrm{SU}(2)$ SYMQM in $d = 1$. In this model, Q does not connect the sectors $n_F = 1$ and $n_F = 2$ (due to the parity of $n_B + n_F$), therefore SUSY properties can be studied separately for the subspaces $n_F \leq 1$ and $n_F \geq 2$; we will consider the reduced Witten index $I_W^{0,1}(t)$, where the sum is restricted to the sectors $n_F = 0$ and $n_F = 1$.

In the present case, $I_W^{0,1}(t)$ can be computed analytically. We transform to polar coordinates and introduce a gauge-invariant infrared regulator R:

$$r = \sqrt{x_a x_a}; \qquad r \leq R.$$

H is free; its eigenfunctions are the spherical harmonics: $\Psi_{p,l}(r) = j_l(pr)$ and $E = \frac{1}{2}p^2$. Gauge invariance implies $J = 0$, therefore $l = 0$ for $n_F = 0$ and $l = 1$ for $n_F = 1$. The allowed values of p are $z_i^{(l)}/R$, where $z_i^{(l)}$ is the ith positive zero of the spherical harmonic j_l. Therefore,

$$I_W^{0,1}(R,t) = \sum_i \left\{ \exp\left[-\frac{t}{2R^2} \left(z_i^{(0)} \right)^2 \right] - \exp\left[-\frac{t}{2R^2} \left(z_i^{(1)} \right)^2 \right] \right\}.$$

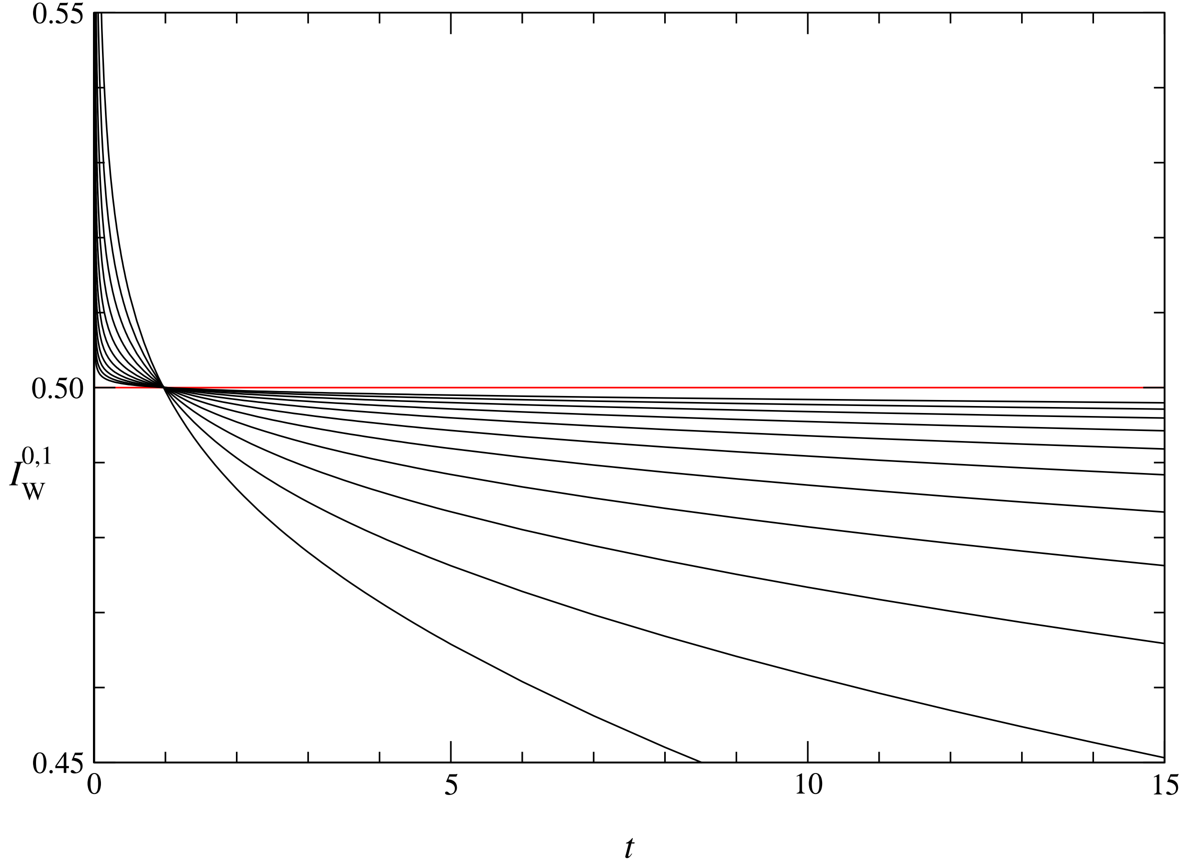
Using the asymptotic form of $z_i^{(l)}$ for large i,

$$z_i^{(0)} = \pi i, \qquad z_i^{(1)} = \beta_i - \frac{1}{\beta_i} + O\left(\frac{1}{i^3}\right), \qquad \beta_i = \pi(i - \tfrac{1}{2}),$$

and replacing the sum with an integral, we obtain

$$\lim_{R\to\infty} I_W^{0,1}(R,t) = \tfrac{1}{2}.$$

$I_W^{0,1}(t)$ can also be computed numerically from the spectrum of H; I show plots for $n_{B\,\max}$ ranging from 125 to 128000; the agreement with the exact value is excellent. At finite $n_{B\,\max}$, $I_W^{0,1}(t) \to 0$ as $t \to 0$; this is verified numerically, but can only be seen in the range $t < 0.02$.



Let us now study $\mathrm{SU}(2)$ SYMQM in $d = 3$; n_F is still conserved; here $n_F \leq 6$ and, thanks to the particle-hole symmetry, it is sufficient to study the sectors $n_F = 0, \ldots, 3$.

The spectrum is discrete in the sectors $n_F = 0,\ 1$; its is a superposition of continuum and discrete states for $n_F = 2,\ 3$.

The computation is similar to the $d = 1$ case; however, it is impossible to obtain closed formulae for the matrix elements; we use instead recursive relations between reduced matrix elements.

It is crucial to exploit fully the $\mathrm{O}(3)$ symmetry; at every step of the computation, only operators with well-defined j, m are considered and, using the Wigner-Eckhart theorem and $6j$ symbols, ms are eliminated from the computation.

A complete basis of gauge-invariant operators needs ca. 75 multiplets; (anti)commutators are computed automatically and stored in a table.

We build states with ever increasing n_F, n_B applying creator operators $X(\nu,p)^\dagger$ in all possible ways and performing Gram-Schmidt orthonormalization:

$$|j,m,n_F,n_B;i\rangle = \sum_{\nu,p,j_1,j_2,j,m_1,m_2} R^{j,n_F,n_B}_{i;\nu,p,j_1,j_2,j}\, C^{j_1\,j_2\,j}_{m_1 m_2 m}\, X(\nu,p)^\dagger_{j_1,m_1}\, |j_2,m_2,n_F{-}\nu,n_B{-}2{-}p{+}\nu;i\rangle,$$

A typical recursion relation for reduced matrix elements is

$$\begin{aligned}
\langle j',n'_F,n'_B;i'\|\mathcal{O}_{j''}\|j,n_F,n_B;i\rangle &= \mp \sum_{\nu,p,j_1,j_2,j;j_3,i_3} (-1)^{j+j''+j_1+j_3}\sqrt{2j+1}\begin{Bmatrix} j & j'' & j' \\ j_3 & j_1 & j_2 \end{Bmatrix} \\
&\times \quad R^{j,n_F,n_B}_{i;\nu,p,j_1,j_2,j}\,\langle j',n'_F,n'_B;i'\|X(\nu,p)^\dagger_{j_1}\|j_3,n'_F{-}\nu,n'_B{-}2{-}p{+}\nu;i_3\rangle \\
&\times \quad \langle j_3,n'_F{-}\nu,n'_B{-}2{-}p{+}\nu;i_3\|\mathcal{O}_{j''}\|j_2,n_F{-}\nu,n_B{-}2{-}p{+}\nu;j\rangle \\
&+ \sum_{\nu,p,j_1,j_2,j;j_3} (-1)^{j'+j''+j_1+j_2}\sqrt{(2j+1)(2j_3+1)}\begin{Bmatrix} j & j'' & j' \\ j_3 & j_2 & j_1 \end{Bmatrix} R^{j,n_F,n_B}_{i;\nu,p,j_1,j_2,j} \\
&\times \quad \langle j',n'_F,n'_B;i'\|K^{(\mathcal{O},j'';\nu,p,j_1)}_{j_3}\|j_2,n_F{-}\nu,n_B{-}2{-}p{+}\nu;j\rangle,
\end{aligned}$$

where we used completeness and the knowledge of the (anti)commutator

$$\{\mathcal{O}_{j_1,m_1},X(\nu,p)^\dagger_{j_2,m_2}\}_\pm = \sum_{j_3,m_3} C^{j_1\,j_2\,j_3}_{m_1 m_2 m_3}\, K^{(\mathcal{O},j_1;\nu,p,j_2)}_{j_3,m_3}.$$

We implemented the recursive computation of reduced matrix elements in a C++ program; once computed, they are kept in RAM, since they will be needed again many times as the computation proceeds.

Running on a 2 GHz AMD Opteron processor, using a total of about 75 hours and 8 Gbytes of RAM, we reached

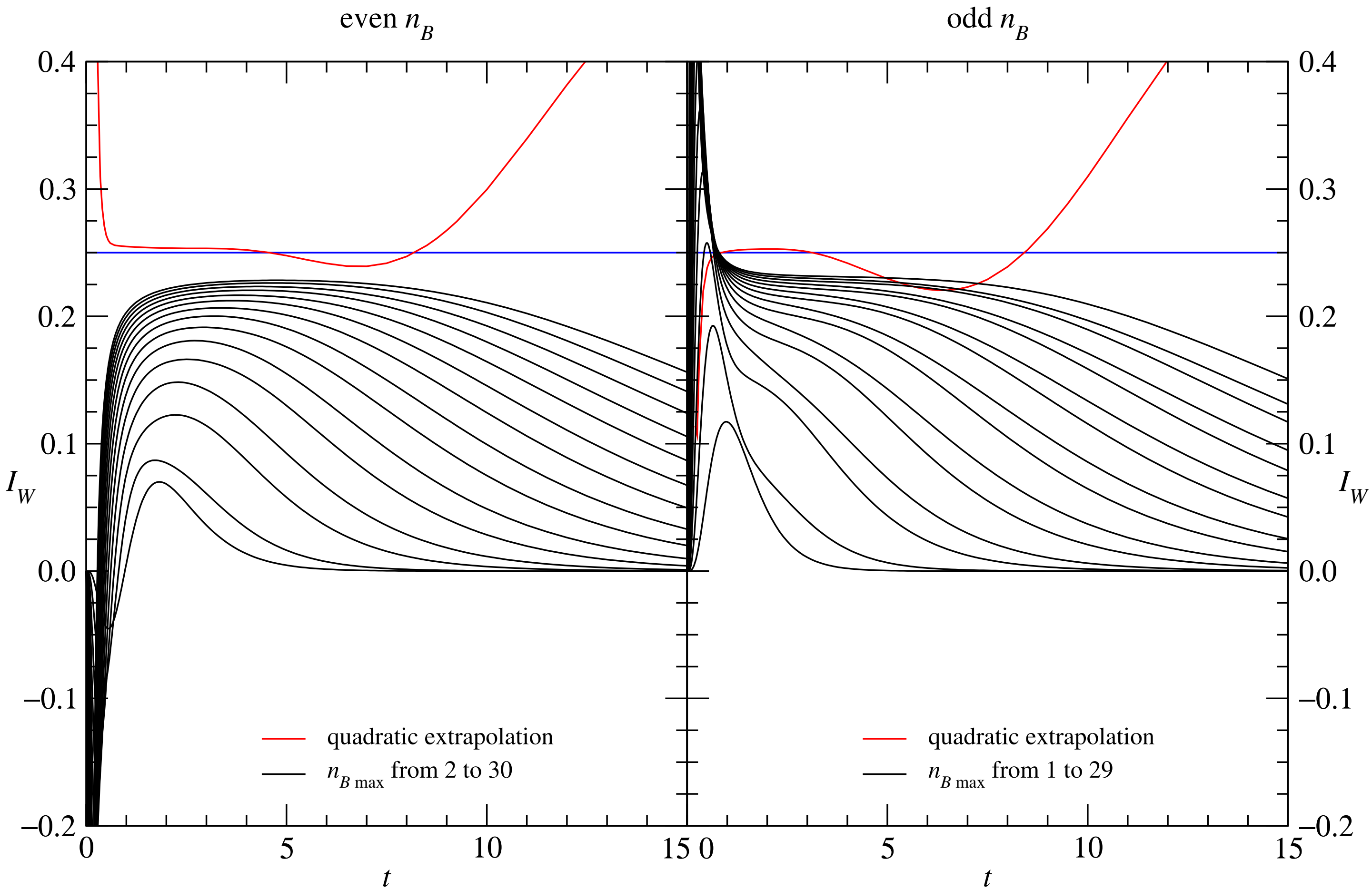
$n_F = 0$: $n_{B\,\mathrm{max}} \leq 61$ ($109\,538$ multiplets, $3\,570\,952$ states)

$n_F = 1$: $n_{B\,\mathrm{max}} \leq 40$ ($94\,688$ multiplets, $2\,125\,200$ states)

$n_F = 2$: $n_{B\,\mathrm{max}} \leq 32$ ($87\,957$ multiplets, $1\,617\,261$ states)

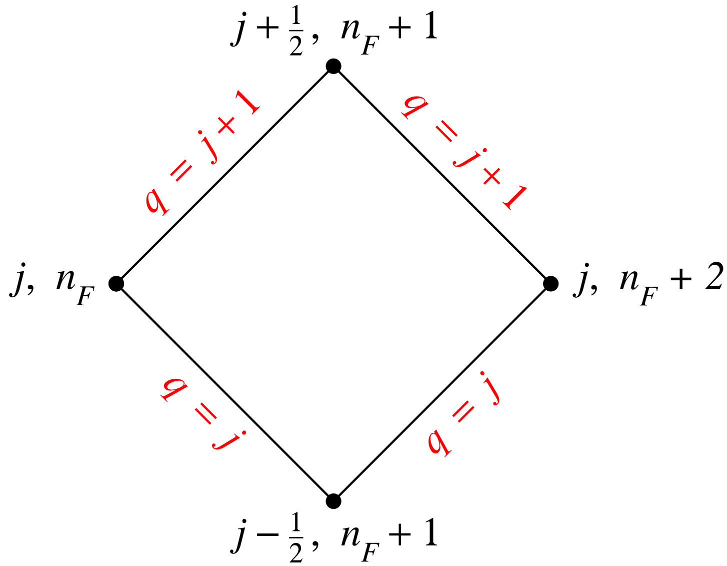
$n_F = 3$: $n_{B\,\mathrm{max}} \leq 30$ ($87\,706$ multiplets, $1\,541\,424$ states)

I show $I_W(t)$ separately for even and odd $n_{B\,\mathrm{max}} \leq 30$, together with a quadratic extrapolation in $1/n_{B\,\mathrm{max}}$ on the 6 largest values; the convergence to $I_W(t) = \frac{1}{4}$ is very clear



Supersymmetric charges $Q^\dagger_\alpha$, Q_α have $n_F = \pm 1$, $j = \frac{1}{2}$ and $m = \pm\frac{1}{2}$; a supermultiplet is composed by 4 $\mathrm{O}(3)$ multiplets (3 for the scalar supermultiplet). In order to classify discrete states, we can search for these patterns among our levels.

We can also look at the "supersymmetry fraction"



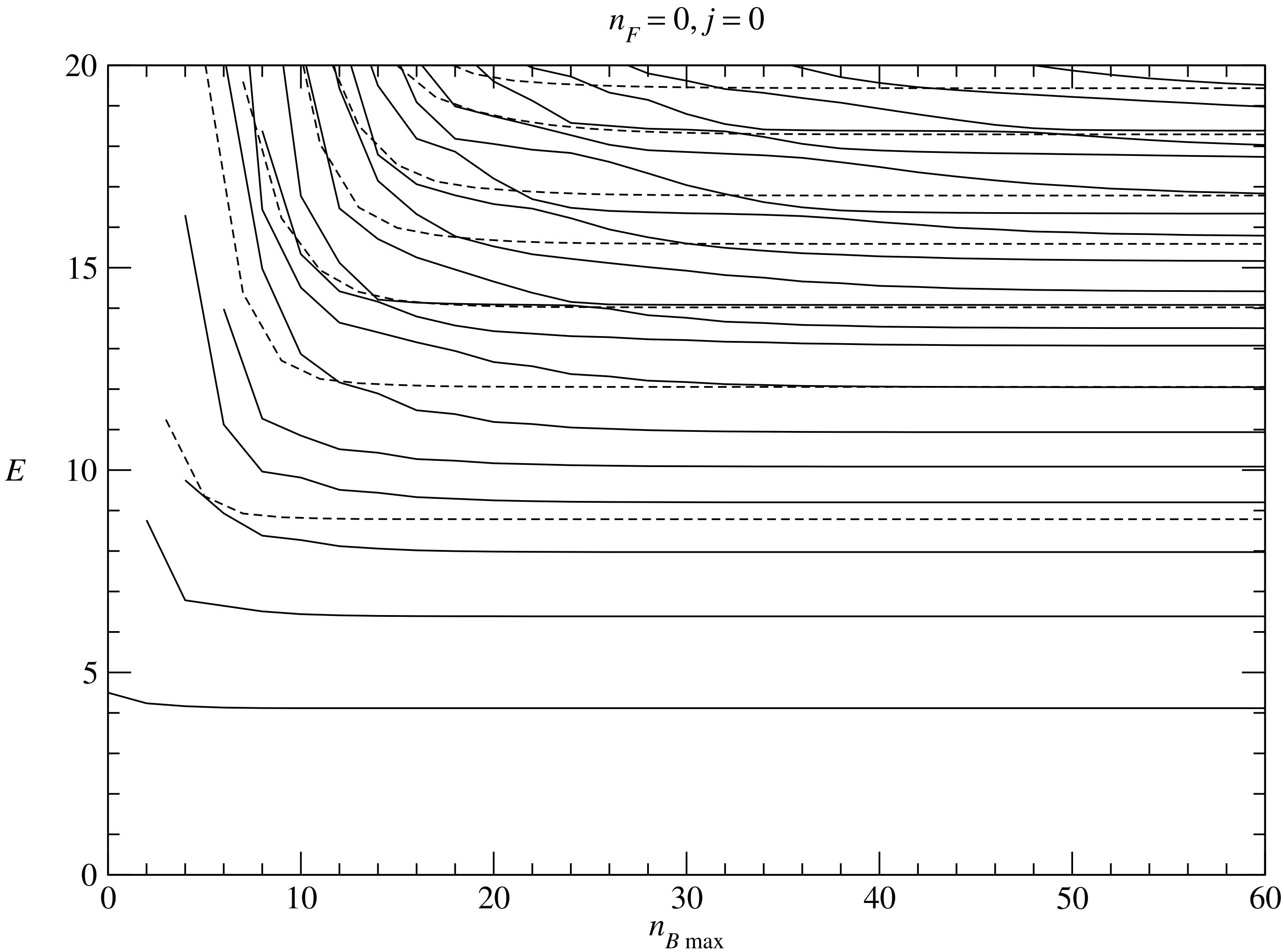
$$q(j', n_F{+}1, i'|j, n_F, i) \equiv \frac{1}{4E_{j,n_F,i}} \left|\langle j'; n_F{+}1; i' \| Q^\dagger \| j; n_F; i\rangle\right|^2,$$

which satisfies the sum rule

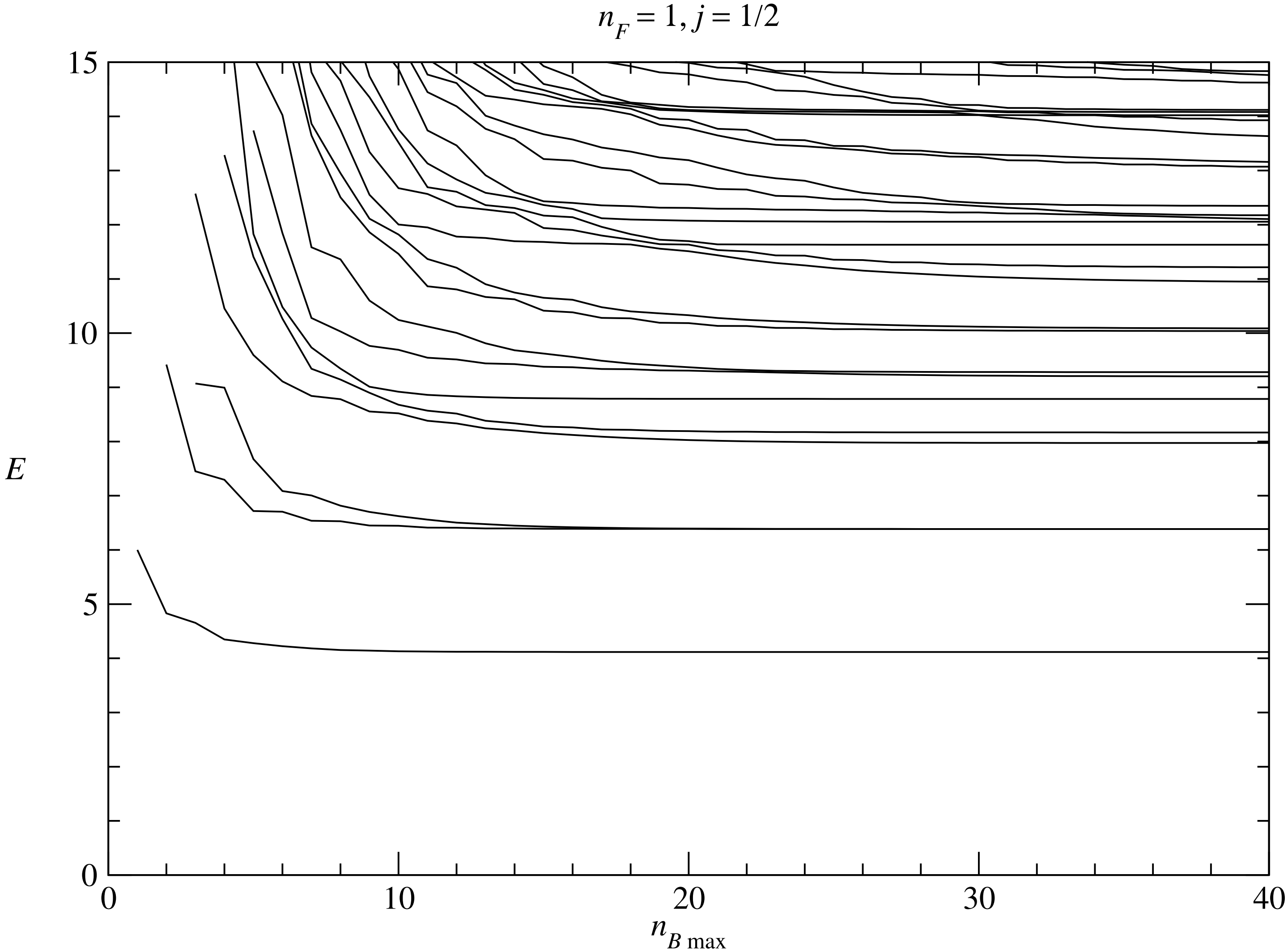
$$\sum_{j',i'} [q(j', n_F{+}1, i'|j, n_F, i) + q(j, n_F, i|j', n_F{+}1, i')] = 2j + 1.$$

For discrete states, it is saturated by one (or very few) states $|j'; n_F{+}1; i'\rangle$ with energy $E_{j',n_F+1,i'} \cong E_{j,n_F,i}$.

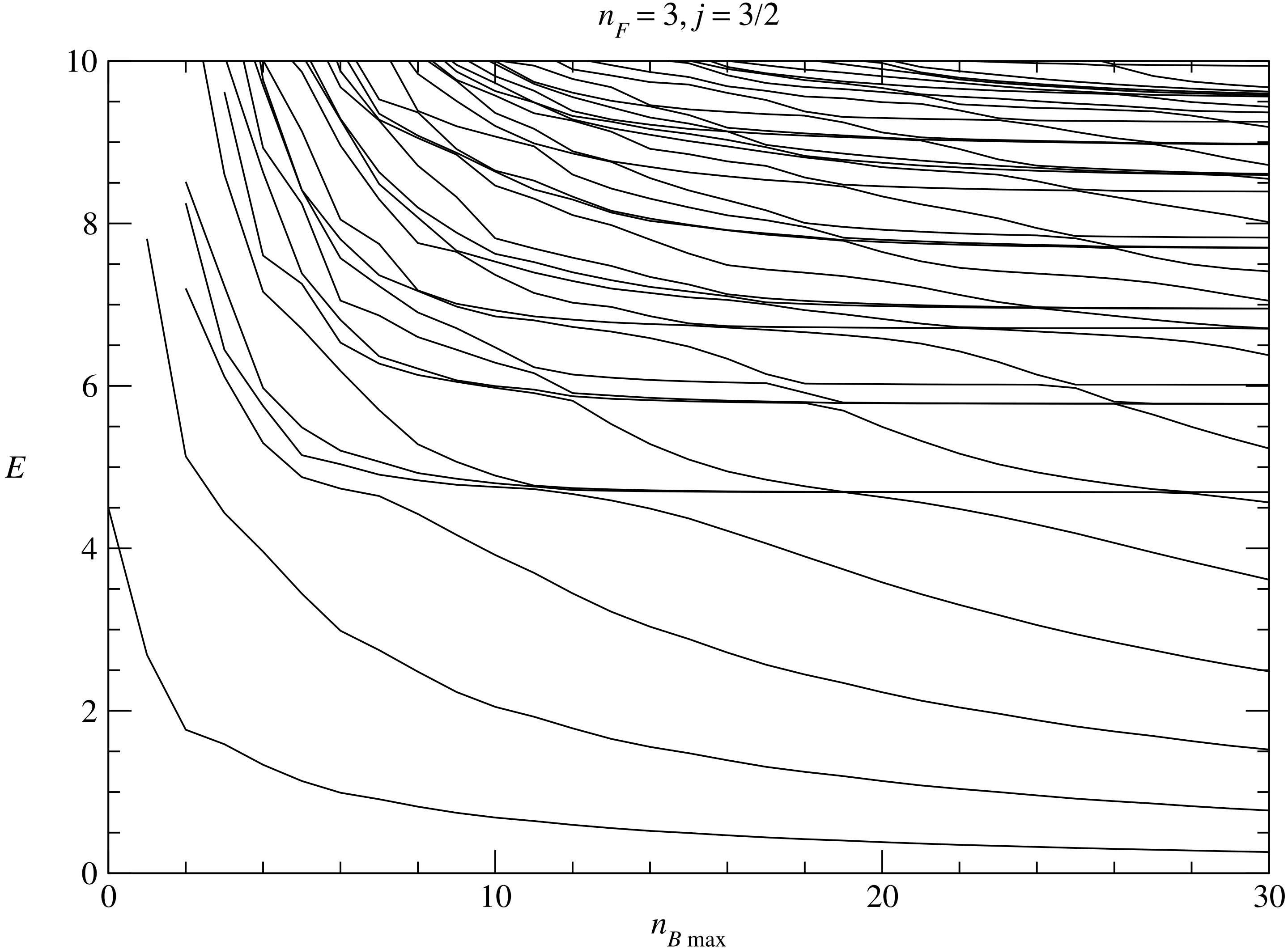
A simple spectrum, all states belonging to the $j = 0, n_F = 0$ supermultiplet:



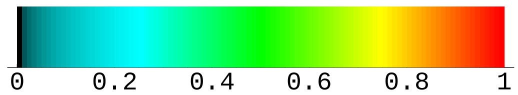
States belonging to either $j = 0, n_F = 0$ or $j = \frac{1}{2}, n_F = 1$



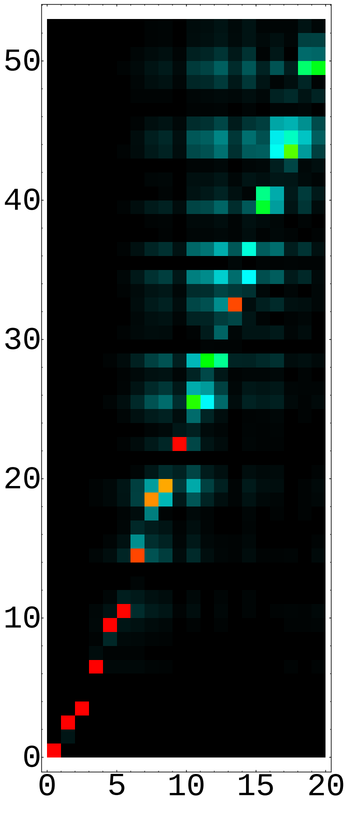
A complicated spectrum: continuum plus 4 supermultiplets, degeneracy due to particle-hole symmetry



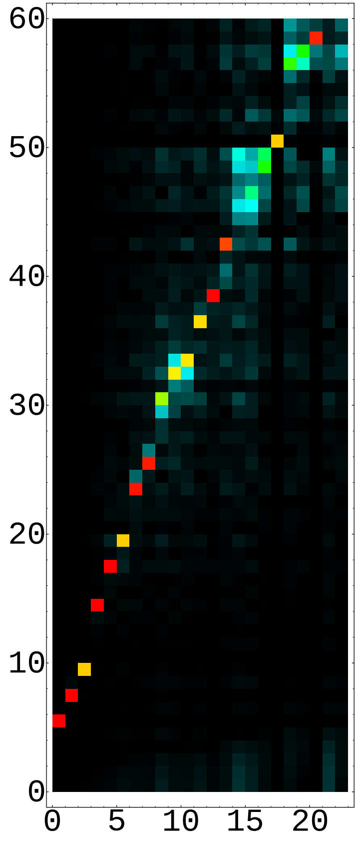
Density plots of $q(j', n_F{+}1, i' | j, n_F, i)/(2\max(j, j') + 1)$.



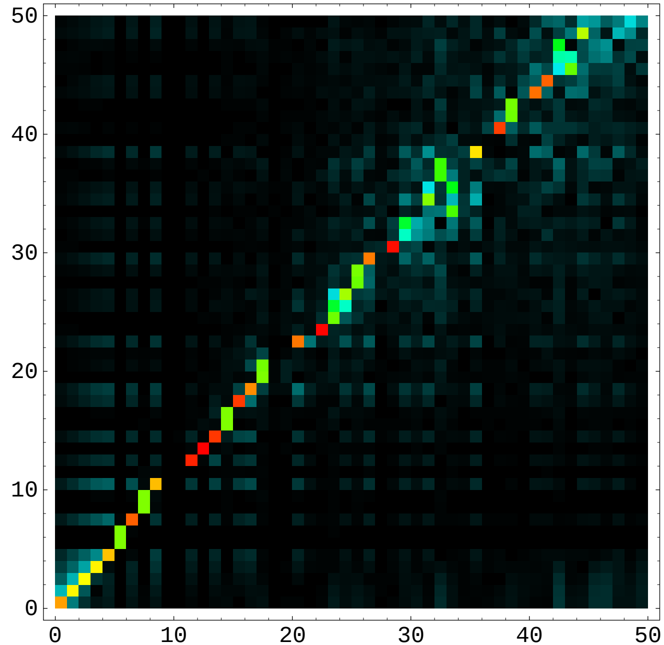
$q(\frac{1}{2}, 1 | 0, 0)$



$q(2, 2 | 1, \frac{3}{2})$



$q(3, \frac{3}{2} | 2, 2)$



$10^3 \times$energies				
$n_F(j)$	(n_F, j)	$(n_F+1, j-\frac{1}{2})$	$(n_F+1, j+\frac{1}{2})$	(n_F+2, j)
$0(0^+)$	4117	—	4117	4117
$0(0^+)'$	6386	—	6386.3	6388
$0(0^+)''$	7973	—	7974	7988
$0(0^+)_{9202}$	9202	—	9204	9254
$0(0^+)_{10086}$	10086	—	10091	10190
$0(0^+)_{10937}$	10937	—	10957	11206
$0(0^+)_{12049}$	12049	—	12128	12720
$0(0^-)$	8787	—	8787	8787
$0(0^-)'$	12055	—	12055	12057
$0(0^-)''$	14020	—	14020	14031
$0(0^-)_{15590}$	15590	—	15590	15624
$2(0)$	5184	—	5184	5184
$2(0)'$	7363	—	7366	7363
$2(0)''$	9875	—	9876	9875
$1(1/2)$	6386	6386	6386	6388, 6389
$1(1/2)'$	8167	8174	8170	8203, 8221
$1(1/2)''$	9281	9298	9288	9337, 9355
$1(1/2)_{10040}$	10040	10085	10077	10251, 10358
$1(1/2)_{11226}$	11226	11395	11349	11728, 11834

$n_F(j)$	(n_F, j)	$(n_F{+}1, j{-}\frac{1}{2})$	$(n_F{+}1, j{+}\frac{1}{2})$	$(n_F{+}2, j)$
$2(1)$	6015	6015	6015	6015
$2(1)'$	7822	7822	7839	7822
$2(1)''$	9350	9344	9396	9350
$2(1)_{9934}$	9934	9932	9956	9934
$1(3/2)$	4692	4692	4692	4691.9, 4692
$1(3/2)'$	5780	5780	5780	5780, 5781
$1(3/2)''$	6950	6951	6952	6955, 6960
$1(3/2)_{7695}$	7695	7696	7899	7709, 7720
$1(3/2)_{8583}$	8583	8587	8596	8623, 8635
$1(3/2)_{8964}$	8964	8967	8973	8989, 8997
$0(2^+)$	6014	6015	6015	6015
$0(2^+)'$	7821	7821	7821	7832
$0(2^+)''$	9332	9334	9334	9406
$0(2^+)_{9928}$	9928	9928	9929	9949
$0(2^-)$	11331	11331	11331	11332
$0(2^-)'$	13998	13998	13998	14010
$0(2^-)''$	15399	15399	15399	15407
$2(2)$	6710	6710	6711	6710
$2(2)'$	8398	8402	8410	8398
$2(2)''$	9255	9259	9276	9255

The techniques and codes we developed are adequate to study $\mathrm{SU}(2)$ SYMQM in $d = 3$ in great detail. We have a good knowledge of the discrete spectrum, and we are ready to investigate the continuum (e.g., defining scattering).

The study of $\mathrm{SU}(N)$ SYMQM with $N \geq 3$ requires additional theoretical work, which is under way [J. Trzetrzelewski, J. Wosiek]. Computing analytically the $N \to \infty$ limit is not out of question.

The most interesting case $d = 9$ requires an algorithm to compute $3j$ and $6j$ coefficients for $\mathrm{O}(9)$: this problem is solved in principle, but in practice we need them for very large representations... we are looking into this [V. Chilla, M. C.].

We think that it is worthwhile to devote a considerable effort to the study of $\mathrm{SU}(N)$ SYMQM in $d = 9$.