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Fields for Trees and Forests

with

Jesper Lykke Jacobsen

Hubert Saleur

Alan D. Sokal

Andrea Sportiello

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Forests and trees

Saguaro cacti forest on hillside, West Unit.
Saguaro National Park, Arizona, USA.

Let *G=(V,E) be a* finite undirected graph with vertex set *V* and edge set *E*.

$$Z_G(q,\mathbf{v}) = \sum_{A\subseteq E} q^{k(A)} \prod_{e\in A} v_e$$

where q and $\mathbf{v} = \{v_e\}_{e\in E}$ are commuting indeterminates, and $k(A)$ denotes the number of connected components in the subgraph *(V,A)*

$$Z_G(q, \mathbf{v})$$

is the multivariate Tutte polynomial of G
alias Potts model (for integer q)
with Boltzmann weight $e^{-\beta H}$ with Hamiltonian

$$H = -\sum_{e \in E} J_e \, \delta\left(\sigma_{x_1(e)}, \sigma_{x_2(e)}\right)$$

$x_{1,2}(e)$ end-points of the edge

$v_e = e^{\beta J_e} - 1 \in [-1, \infty]$ (Fortuin-Kasteleyn)

$\sigma : V \mapsto \{1, 2, \ldots, q\}$

in the limit $q \to 0$ at fixed $\mathbf{w} = \mathbf{v}/q$

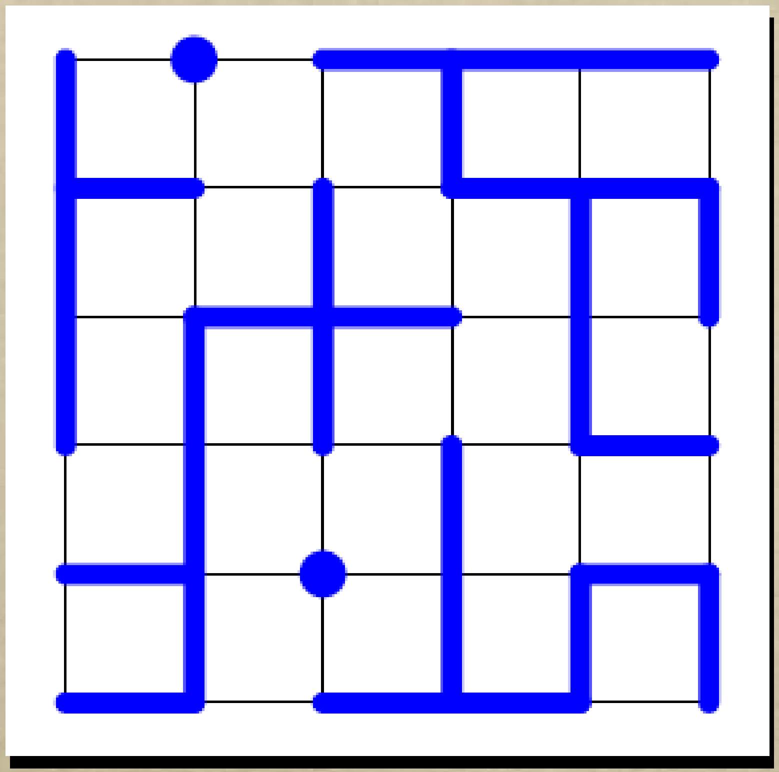
$$\lim_{q \to 0} q^{-|V|} Z_G(q, q\mathbf{w}) = F_G(\mathbf{w})$$

where
$$F_G(\mathbf{w}) = \sum_{\substack{A \subseteq E \\ c(A)=0}} \prod_{e \in A} w_e$$

is the generating polynomial of spanning forests

$$c(A) = k(A) - |V| + |A|$$

is the cyclomatic number (no. of independent cycles)

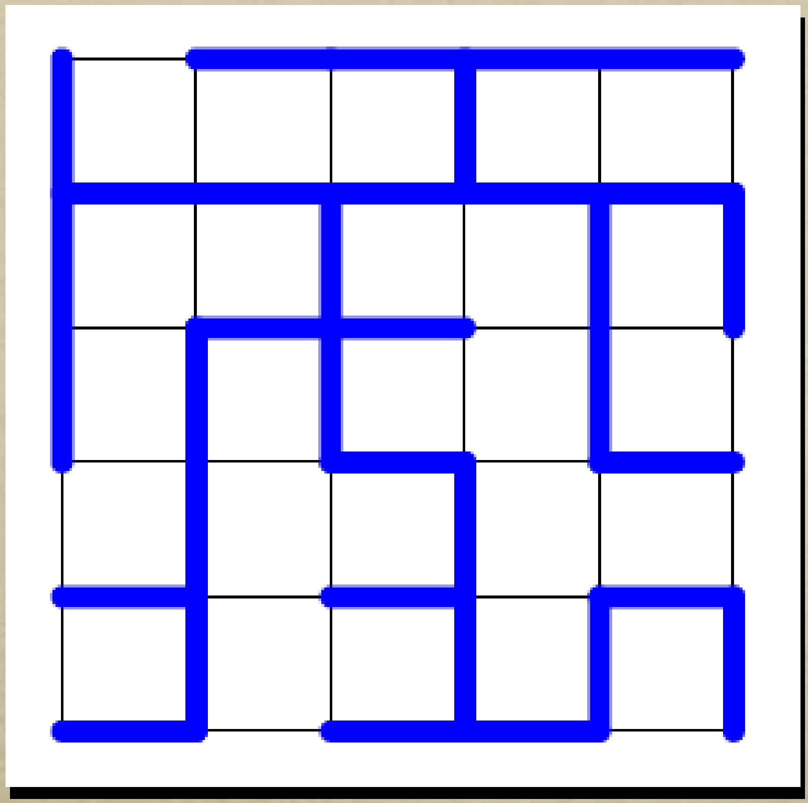
A forest is a subgraph of G if it contains no cycles and is called spanning if its vertex set is exactly V

the limit

$$\lim_{\lambda \to \infty} \lambda^{k(G)-|V|} F_G(\lambda \mathbf{w}) = T_G(\mathbf{w})$$

where $$T_G(\mathbf{w}) = \sum_{\substack{A \subseteq E \\ k(A)=k(G) \\ c(A)=0}} \prod_{e \in A} w_e$$

is the generating polynomial of spanning trees (spanning forests with the maximal number of edges)

A tree is a subgraph of G if it is connected and contains no cycles, and is called spanning if its vertex set is exactly V.

The matrix-tree theorem *(1847)*

Gustav Robert Kirchhoff

(electrical circuit theory)

Ann. Phys. Chem. 72, 497 (1847)

For $i \neq j$ let $w_{ij} = w_{ji}$ be the sum of w_e over all edges e that connect i to j.

The (weighted) Laplacian L matrix for the graph G is defined by

$$L_{ij} = \begin{cases} -w_{ij} & \text{for } i \neq j \\ \sum_{k \neq i} w_{ik} & \text{for } i = j \end{cases}$$

Let $L(i)$ be the matrix obtained from L by deleting the i-th row and column, then

$$\det L(i) = T_G(\mathbf{w})$$

independently from the root i (which in electrical-circuit language is the choice of the ground)

Grass field

Grassmann field

Introduce at each vertex i a pair of Grassmann variables

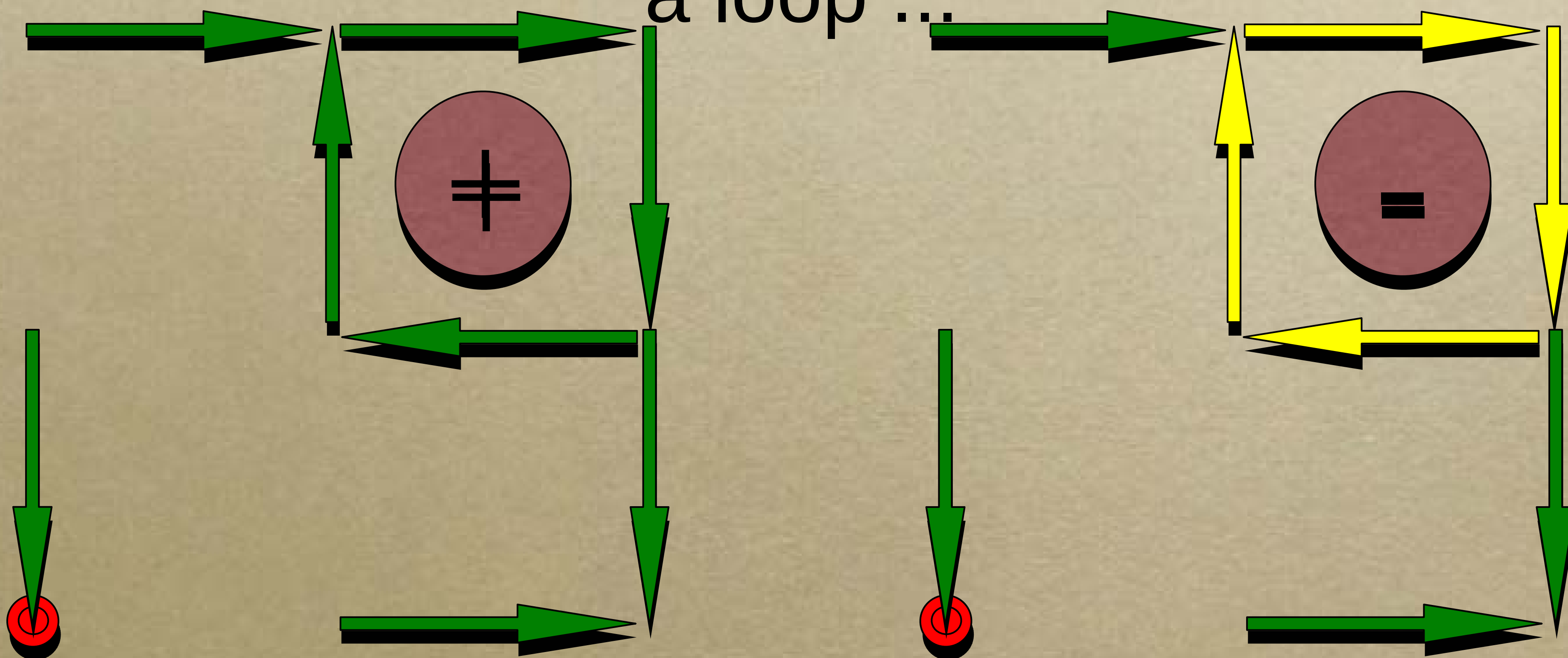
$$\psi_i , \bar{\psi}_i$$

$$\det L(i) = \int \mathcal{D}(\psi, \bar{\psi})\, \bar{\psi}_i \psi_i \, e^{\bar{\psi} L \psi}$$

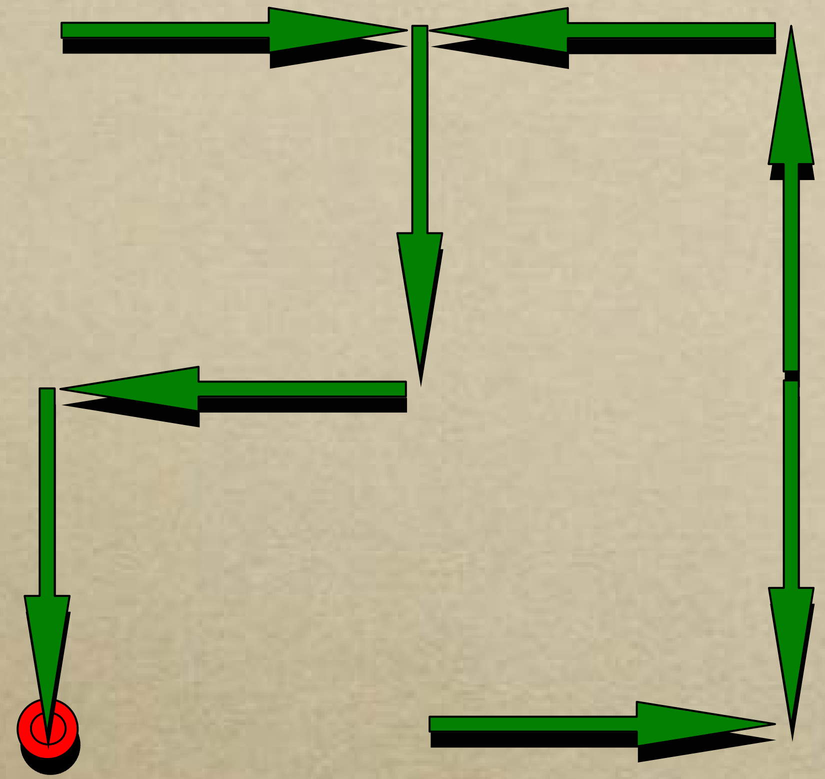
$$\mathcal{D}(\psi, \bar{\psi}) = \prod_{k \in V} d\psi_k \, d\bar{\psi}_k$$

$$\bar{\psi} L \psi = \sum_{i \neq j} \left[-\bar{\psi}_i w_{ij} \psi_j + \bar{\psi}_i w_{ij} \psi_i \right]$$

We must have both fields at each vertex, lines must end in a root by a green line, but if there is a loop ...



$$\bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 \bar{\psi}_3 \psi_3 \bar{\psi}_3 \psi_3 = -\bar{\psi}_1 \psi_2 \bar{\psi}_2 \psi_3 \bar{\psi}_3 \psi_4 \bar{\psi}_4 \psi_1$$



green lines mush finish somewhere: at the root!

Only one connected component can remain, therefore each configuration is a tree.

Principal-minors matrix-tree theorem

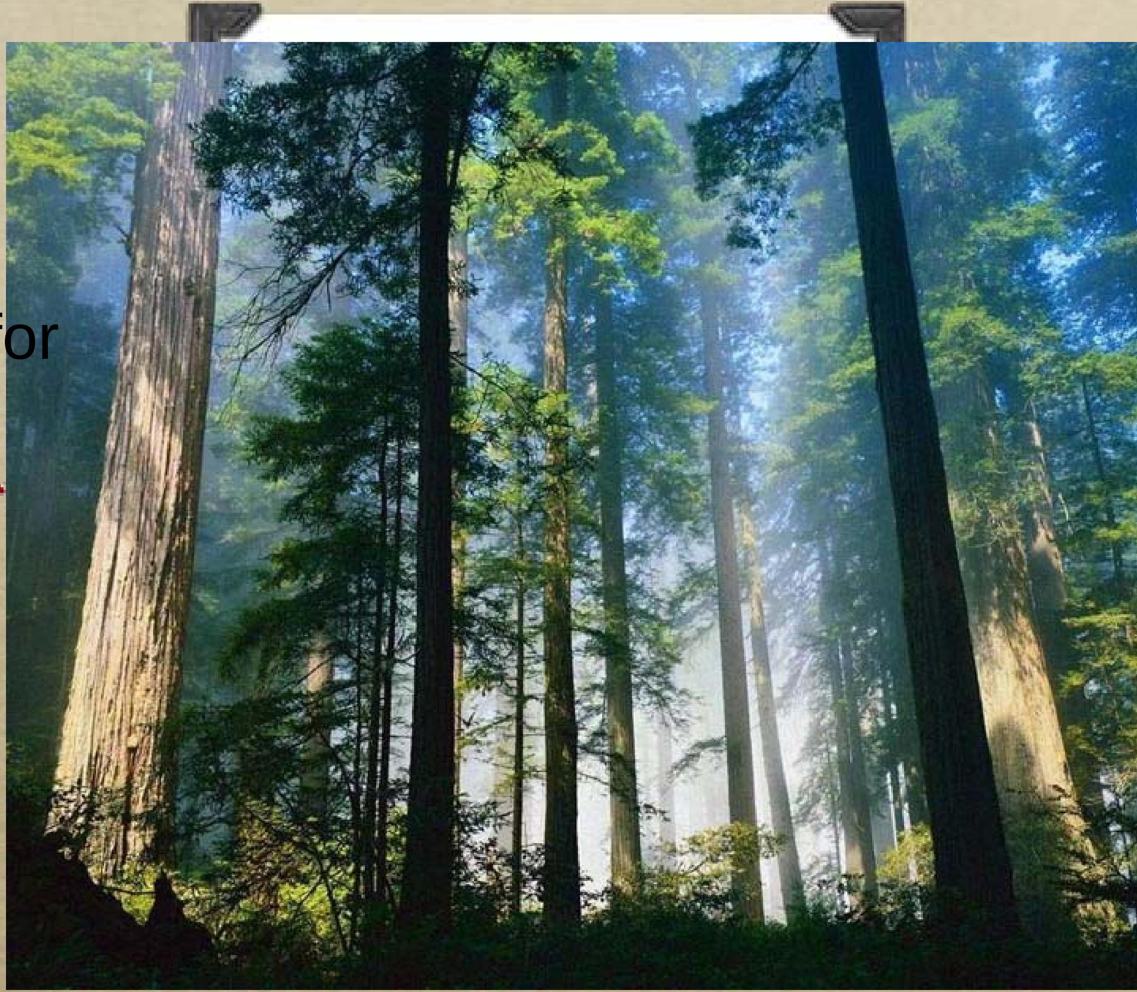
$$\det L(i_1, \ldots, i_r) = \int \mathcal{D}(\psi, \bar{\psi}) \prod_{k=1}^{r} \bar{\psi}_{i_k} \psi_{i_k} \, e^{\bar{\psi} L \psi}$$

$$= \sum_{F \in \mathcal{F}(i_1, \ldots, i_r)} \prod_{e \in F} w_e$$

where the sum runs over all spanning forests composed of r disjoint trees, each of which contains exactly one of the root vertices

$$i_1, \ldots, i_r$$

A Grassmann representation for unrooted spanning forests



Let

$$Q_\Gamma = \left(\prod_{e\in E_\Gamma} w_e\right)\left(\prod_{i\in V_\Gamma} \bar{\psi}_i\psi_i\right)$$

for each connected subgraph $\Gamma = (V_\Gamma, E_\Gamma)$

Let us try to evaluate

$$\int \mathcal{D}(\psi,\bar{\psi})\, Q_{\Gamma_1}\cdots Q_{\Gamma_l}\, e^{\bar{\psi}L\psi}$$

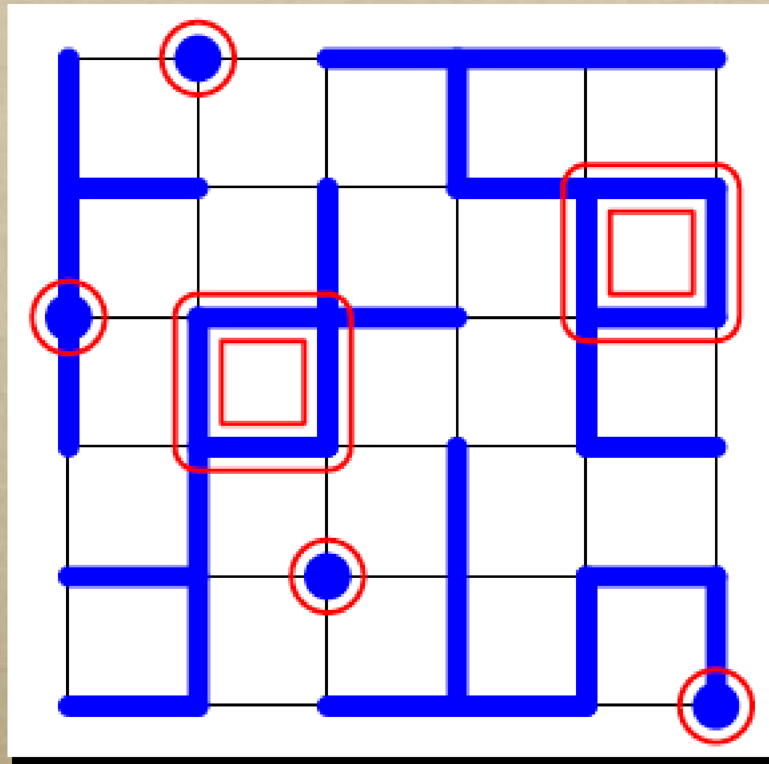
If the subgraphs have one or more vertices in common this integral vanishes

$$\left(\prod_{k=1}^{l}\prod_{e\subset E_{\Gamma_k}} w_e\right)\int \mathcal{D}(\psi,\bar{\psi})\left(\prod_{k=1}^{l}\prod_{i\subset V_{\Gamma_k}}\bar{\psi}_i\psi_i\right)e^{\bar{\psi}L\psi}$$

The integral can be written as a sum over forests rooted at the vertices of $V_{\Gamma} = \bigcup_{k=1}^{l} V_{\Gamma_k}$

All the edges of $E_{\Gamma} = \bigcup_{k=1}^{l} E_{\Gamma_k}$ must be absent from these forests, since otherwise two or more of the root vertices would lie in the same component.

On the other hand, by adjoining the edges of E_{Γ} ,
these forests can be put into one-to-one correspondence
with what we shall call Γ -forests



A typical spanning Γ*-forest configuration on a portion of the square lattice.*
Root subgraphs are the single-vertex graph and the square-shaped graph.

Γ- forests are spanning subgraphs in G whose edge set contains E_{Γ} and which, after deletion of the edges in E_{Γ}, leaves a forest in which each tree component contains exactly one vertex from V_{Γ}.

$$\int \mathcal{D}(\psi,\bar\psi)\, Q_{\Gamma_1}\cdots Q_{\Gamma_l}\, e^{\bar\psi L\psi} = \sum_{H\in\mathcal{F}_{\mathbf{\Gamma}}} \prod_{e\in H} w_e$$

where the sum runs over all $\mathbf{\Gamma}$ -forests *H*. So that as

$$1 + t_\Gamma Q_\Gamma = e^{t_\Gamma Q_\Gamma}$$

$$\int \mathcal{D}(\psi,\bar\psi)\, e^{\bar\psi L\psi + \sum_\Gamma t_\Gamma Q_\Gamma} = \sum_{\mathbf{\Gamma}} \Big(\prod_{\Gamma\in\mathbf{\Gamma}} t_\Gamma\Big) \sum_{H\in\mathcal{F}_{\mathbf{\Gamma}}} \prod_{e\in H} w_e$$

$$= \sum_{\substack{H \text{spanning } \subseteq G \\ H=\{H_1,\dots,H_l\}}} \left(\prod_{i=1}^{l} W(H_i)\right) \prod_{e\in H} w_e$$

where $W(H_i) = \sum_{\Gamma\prec H_i} t_\Gamma$

Main result

$$\int \mathcal{D}(\psi,\bar\psi)\ \exp\left[\bar\psi L\psi + t\sum_i \bar\psi_i\psi_i + u\sum_{\langle ij\rangle} w_{ij}\bar\psi_i\psi_i\bar\psi_j\psi_j\right]$$

$$= \sum_{\substack{F\,\in\,\mathcal{F}\\ F\,=\,(F_1,\ldots,F_l)}} \left(\prod_{i=1}^{l}\left(t|V_{F_i}| + u|E_{F_i}|\right)\right)\prod_{e\in F} w_e$$

Application: *u=-t*

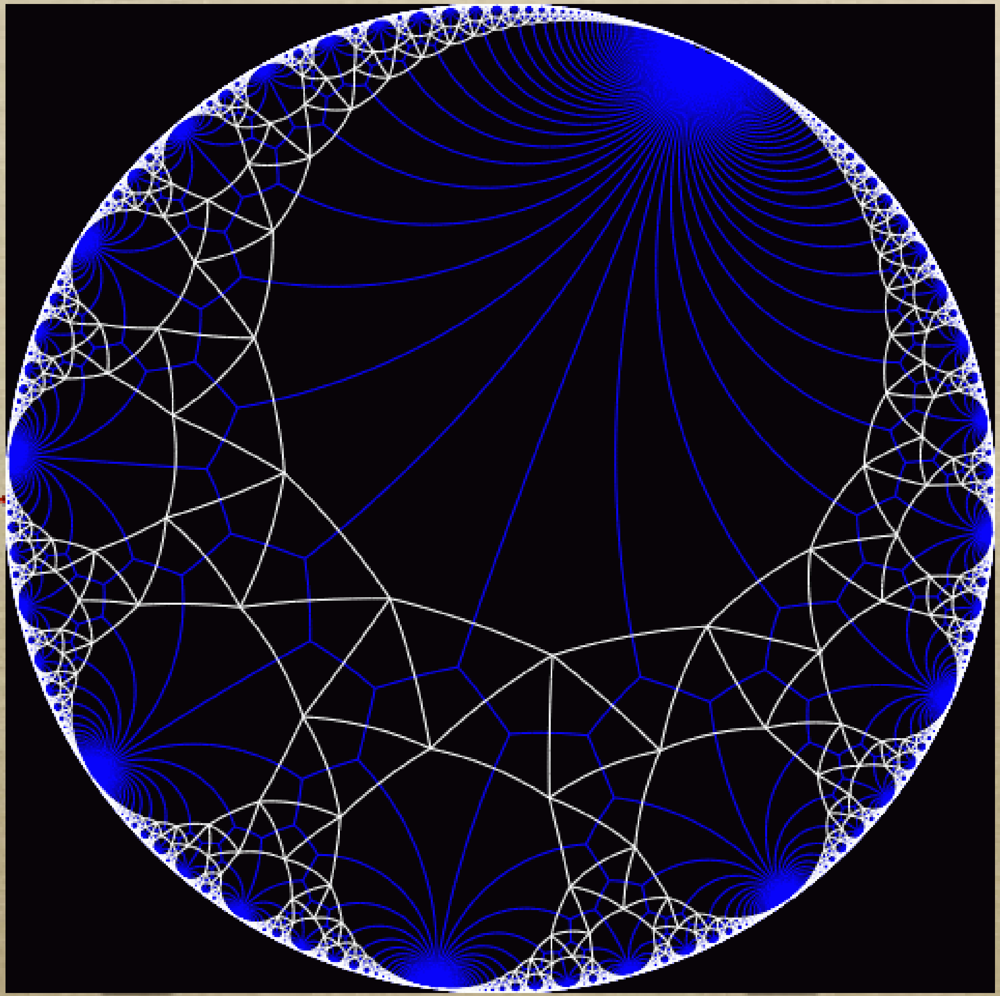
since for each tree $|V_{F_i}| - |E_{F_i}| = 1$

$$\int \mathcal{D}(\psi,\bar{\psi})\, \exp\Big[\bar{\psi}L\psi \,+\, t\sum_i \bar{\psi}_i\psi_i \,-\, t\sum_{\langle ij\rangle} w_{ij}\bar{\psi}_i\psi_i\bar{\psi}_j\psi_j\Big]$$

$$= \, t^{|V|}\, F_G(\mathbf{w}/t)$$

we obtain the generating function of unrooted spanning forests with a weight t for each component

Is there a symmetry to fix this coupling constant relation?



Mapping onto Lattice σ-Models

Recall that the N-vector model consists of spins

$$\boldsymbol{\sigma}_i \in \mathbb{R}^N \quad i \in V \quad |\boldsymbol{\sigma}_i|^2 = 1$$

with Boltzmann weight $e^{-\mathcal{H}}$ where

$$\mathcal{H} = -T^{-1} \sum_{\langle ij \rangle} w_{ij} (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j - 1)$$

T is the temperature

Low-temperature perturbation theory is obtained by writing (and expanding in powers of $\boldsymbol{\pi}$)

$$\boldsymbol{\sigma}_i = \left(\sqrt{1 - T\boldsymbol{\pi}_i^2}, T^{1/2}\boldsymbol{\pi}_i \right)$$

Taking into account the Jacobian, the effective Hamiltonian is

$$\mathcal{H}' = \frac{1}{2}\sum_{i,j} L_{ij}\boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_j - \frac{T}{2}\sum_i \boldsymbol{\pi}_i^2 - \frac{T}{4}\sum_{\langle ij \rangle} w_{ij}\boldsymbol{\pi}_i^2\boldsymbol{\pi}_j^2 + O(\boldsymbol{\pi}_i^4, \boldsymbol{\pi}_j^4)$$

When $N=-1$, the bosonic field $\boldsymbol{\pi}$ has -2 components,
and so can be replaced by a Grassmann pair if we make the substitution

$$\boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_j \to \psi_i \bar{\psi}_j - \bar{\psi}_i \psi_j$$

Higher powers of $\boldsymbol{\pi}$ vanish due to the nilpotence of the Grassmann fields and we obtain the spanning-forest if we identify $t=-T$, $u=T$ (anti-ferromagnetic N-vector model)

Supersimmetric formulation

An alternate mapping can be obtained by introducing at each site, the superfield

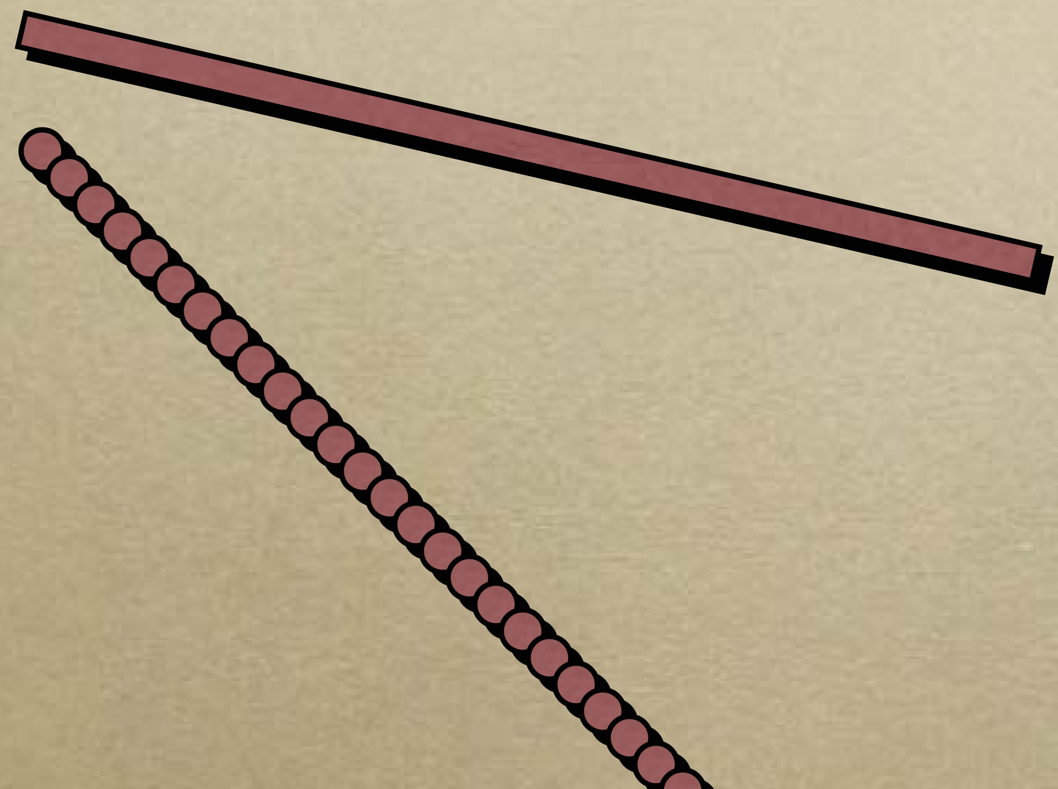
$$\vec{\boldsymbol{\sigma}}_i = (\varphi_i, \psi_i, \bar{\psi}_i)$$

with scalar product

$$\vec{\boldsymbol{\sigma}}_i \cdot \vec{\boldsymbol{\sigma}}_j = \varphi_i \varphi_j + t(\bar{\psi}_i \psi_j - \psi_i \bar{\psi}_j)$$

This corresponding σ -model, is invariant under the supergroup *OSP(1|2)*

Continuum limit



Suppose that the graph *G* is a regular *2-D* lattice with weight w for each nearest-neighbor pair.

We can then read off,

from known results on the *N*-vector model

the RG flow for the spanning-forest model on $\mathbf{Z}^2$

$$\frac{dt}{d\ell} = \frac{3}{2\pi}\bar{t}^2 - \frac{3}{(2\pi)^2}\bar{t}^3 + \frac{2.34278457}{(2\pi)^3}\bar{t}^4 + \frac{1.43677}{(2\pi)^4}\bar{t}^5 + \ldots$$

$$\bar{t} = t/w$$

S. Caracciolo and A. Pelissetto: Nuclear Physics B455 (1995) 619

- the positive coefficient of the t^2 term indicates that for $t > 0$ the model is perturbatively asymptotically free. It is attracted to the infinite-temperature fixed point, hence is massive and OSP(1|2)-symmetric

- for $t_{\mathrm{crit}} < t < 0$ the model is attracted to the free-fermion fixed point at t=0, and hence is massless with central charge c=-2, with the OSP(1|2) symmetry spontaneously broken

- for $t < t_{\mathrm{crit}}$ we expect that the model will again be massive, with the OSP(1|2) symmetry restored

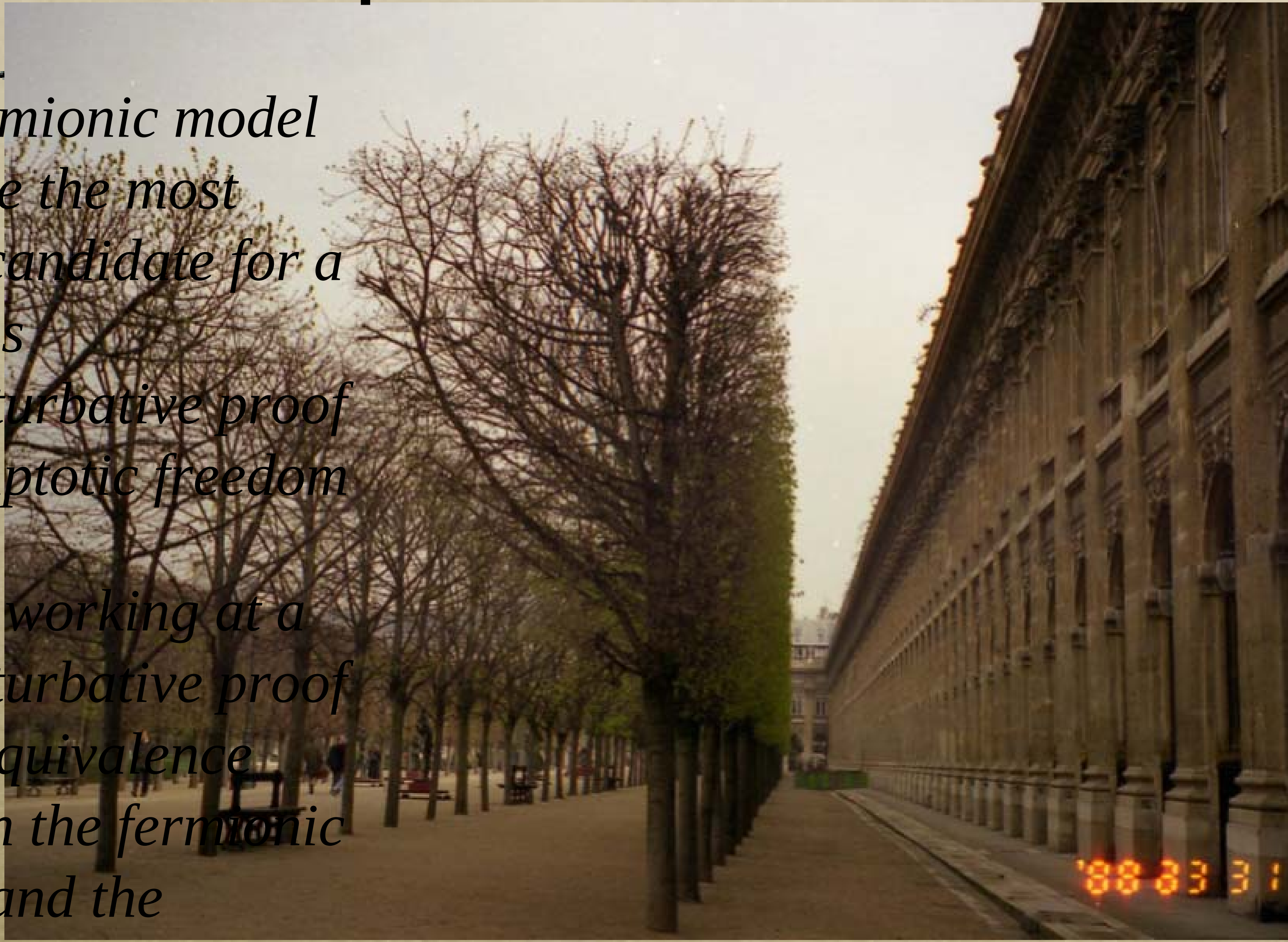
More specifically, for t>0 it is predicted that the correlation length diverges as

$$\xi = C_\xi \, e^{(2\pi/3)(w/t)} \left(\frac{2\pi}{3}\frac{w}{t}\right)^{1/3} \times \left[1 - 0.0116221204\frac{t}{w} + 0.00446142\frac{t^2}{w^2} + \ldots\right]$$

Nmerical results based on transfer matrices and finite-size scaling (for the Potts model), are consistent with the nonperturbative validity of the asymptotic-freedom predictions.

J.-L. Jacobsen, J. Salas, A.D. Sokal, cond-mat/0401026

Perspectives



- *Our fermionic model could be the most viable candidate for a rigorous nonperturbative proof of asymptotic freedom*

- *We are working at a nonperturbative proof of the equivalence between the fermionic theory and the OSP(1/2) sigma model*