XII WORKSHOP ON STATISTICAL MECHANICS

AND NON PERTURBATIVE FIELD THEORY

Bari, September 29 - October 1, 2004

UNIVERSALITY ALONG A LINE OF CRITICAL POINTS. THE ASHKIN-TELLER MODEL

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Based on :

G.D. and P. Grinza, Nucl. Phys. B 682 (2004) 521

Main issues

- Field theoretical predictions for lattice models; universality
- Need for non-perturbative methods; integrability
- Hints for theoretical progress on basic models
- Challenge for lattice people

Outline of the talk

- Universal ratios along a line of critical points
- The Ashkin-Teller model
- Scaling limit and field theory (Kadanoff)
- Results from integrable quantum field theory

Critical points vs critical lines

Example : take the susceptibility χ

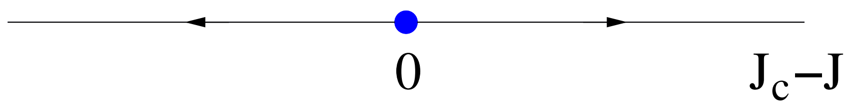
$$\lim_{t \to 0^{\pm}} \chi = \Gamma_{\pm} |t|^{-\gamma} \tag{1}$$

γ is universal

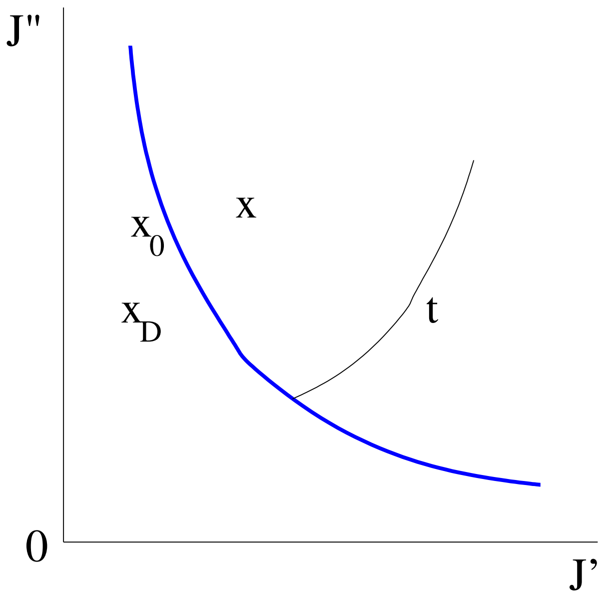
$\Gamma_{\pm}$ are not universal, but Γ_{+}/Γ_{-} is

- **Isolated critical point :**

$$t \sim J_c - J$$



- **Line of critical points :**



(1) defines continously varying exponents and amplitudes with t coordinate along an *unknown* path

$\Gamma_{\pm}$ are not measurable, but their ratio is:

$$\frac{\Gamma_{+}}{\Gamma_{-}}(x_0) = \lim_{x \to x_0} \frac{\chi(x)}{\chi(x_D)}$$

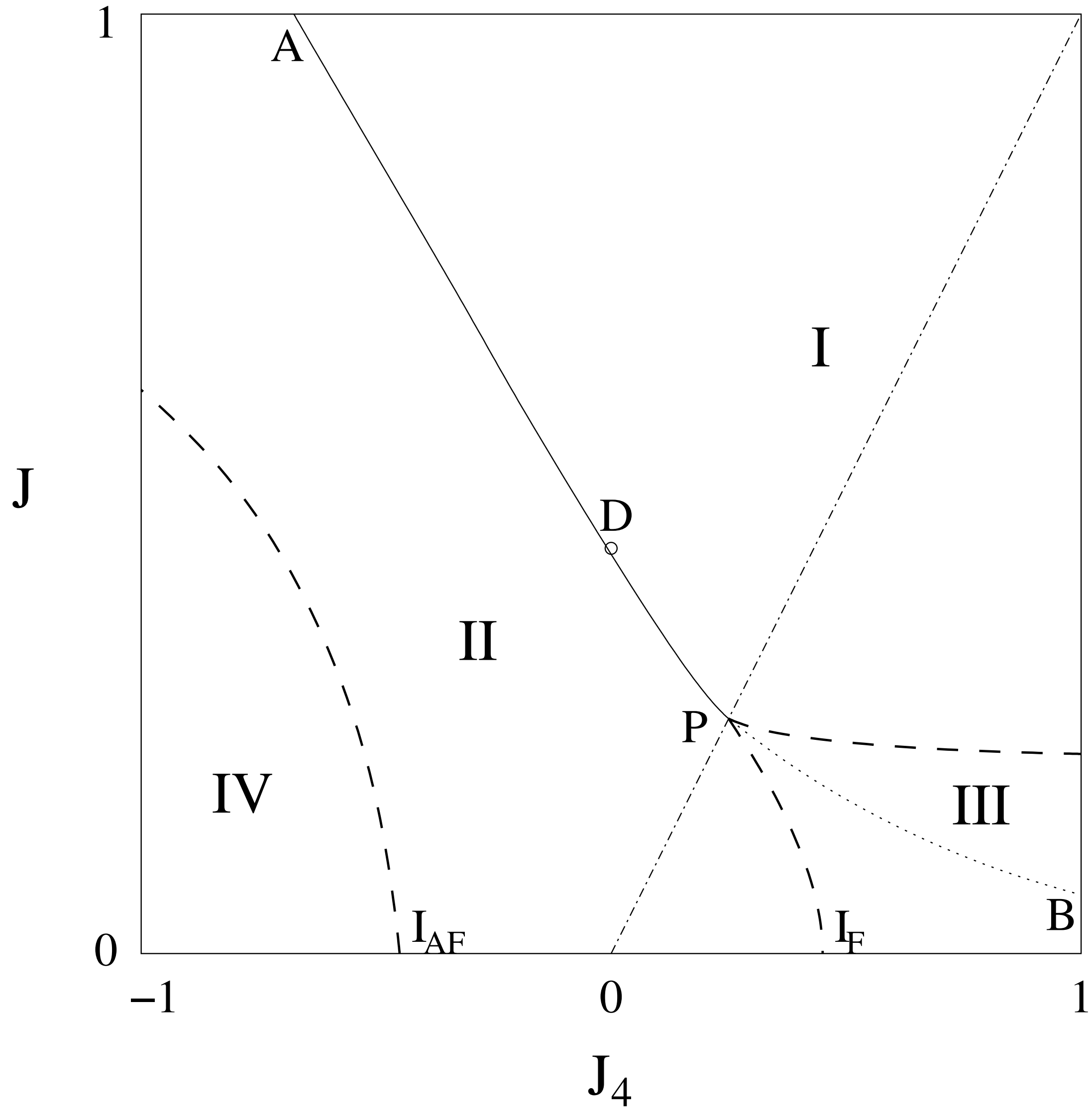
$x_D =$ dual of x

The isotropic Ashkin-Teller model

$$H = -\sum_{\langle xy \rangle} \{ J[\sigma_1(x)\sigma_1(y) + \sigma_2(x)\sigma_2(y)] + J_4\, \sigma_1(x)\sigma_1(y)\sigma_2(x)\sigma_2(y) \}$$

$$\sigma_1, \sigma_2 = \pm 1$$

- **Symmetries :**
  - Spin reversal ($Z_2 \times Z_2$) : $\sigma_1 \to -\sigma_1, \quad \sigma_2 \to -\sigma_2$
  - Exchange : $\sigma_1 \leftrightarrow \sigma_2$
- **Phase diagram on the square lattice :**



Reflection symmetry about the J_4 axis

- **Special cases :**

– $J_4 = 0$: decoupled Ising subspace

– $J = J_4$: 4-state Potts subspace

(symmetric role for σ_1, σ_2 and $\sigma_1\sigma_2$)

– $J = 0$: Ising model in $\sigma_1\sigma_2$

– $J_4 \to +\infty$: $\sigma_1(x)\sigma_1(y) = \sigma_2(x)\sigma_2(y)$

$\Rightarrow$ Ising model with coupling $2J$

- **Duality :** J_0 arbitrary constant

$$\begin{aligned} \omega_1 &= e^{2J+J_4+J_0} \\ \omega_2 &= e^{-J_4+J_0} \\ \omega_3 &= e^{-2J+J_4+J_0} \end{aligned} \qquad \begin{aligned} \tilde{\omega}_1 &= (\omega_1 + 2\omega_2 + \omega_3)/2 \\ \tilde{\omega}_2 &= (\omega_1 - \omega_3)/2 \\ \tilde{\omega}_3 &= (\omega_1 - 2\omega_2 + \omega_3)/2 \end{aligned}$$

- **Self-dual line :**

$$\sinh 2J = e^{-2J_4}$$

Baxter : the self-dual line is critical up to the Potts point;

exact solution on the self-dual line;

continously varying critical exponents

- **Away from the self-dual line :**

no exact solution, numerical results;

3 critical lines with fixed (Ising) exponents, 4 phases :

I) ferromagnetic, $\langle\sigma_1\rangle$, $\langle\sigma_2\rangle$, $\langle\sigma_1\sigma_2\rangle \neq 0$

II) paramagnetic, $\langle\sigma_1\rangle = \langle\sigma_2\rangle = \langle\sigma_1\sigma_2\rangle = 0$

III) partially ferromagnetic, $\langle\sigma_1\rangle = \langle\sigma_2\rangle = 0$, $\langle\sigma_1\sigma_2\rangle \neq 0$

IV) partially antiferro, $\langle\sigma_1\rangle = \langle\sigma_2\rangle = 0$, $\langle\sigma_1\sigma_2\rangle \neq 0$

Scaling limit and field theory

$$\mathcal{A}_{scaling} = \mathcal{H}_1^0+\mathcal{H}_2^0+\tau\int d^2x\,[\varepsilon_1(x)+\varepsilon_2(x)]+\rho\int d^2x\,\varepsilon_1\varepsilon_2(x)$$

$\mathcal{H}_j^0$ Ising fixed point Hamiltonians

$\varepsilon_j(x)$ Ising energy operators

$\varepsilon_1\varepsilon_2(x)$ marginal operator

τ, ρ non-universal functions of J, J_4

- **Line of fixed points:**

$\tau = 0$: $\qquad c = \frac{1}{2}+\frac{1}{2} = 1$

Dirac fermion $\psi(x) = \psi_1(x)+i\psi_2(x)$ $\quad(\varepsilon_j = \psi_j^+\psi_j^-)$ **or**

neutral boson $\varphi(x) = \phi(z)+\bar{\phi}(\bar{z})$ $\quad(\varepsilon_1+\varepsilon_2 = \cos\beta\varphi)$

- **Locality:**

$$\langle...\Phi_1(ze^{2i\pi},\bar{z}e^{-2i\pi})\Phi_2(0,0)...\rangle = l_{\Phi_1,\Phi_2}\,\langle...\Phi_1(z,\bar{z})\Phi_2(0,0)...\rangle$$

Φ_1, Φ_2 mutually local if $l_{\Phi_1,\Phi_2} = 1$

Operators local with respect to the energy $\cos\beta\varphi$:

$$\begin{cases} \sigma_j,\ \mu_j & j = 1,2 \\ V_p = \exp[ip\varphi] & \\ U_{n,m} = \exp\left[i\frac{n\beta}{2m}\varphi+i\frac{2\pi m}{\beta}\tilde{\varphi}\right], & n = 0,\pm 1,...,\ m = \pm 1,... \end{cases}$$

$$(\tilde{\varphi} = \phi(z)-\bar{\phi}(\bar{z}),\qquad n = 2\times\text{spin})$$

In particular

$$\psi = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = \begin{pmatrix}\psi_1^+ + i\psi_2^+\\ \psi_1^- + i\psi_2^-\end{pmatrix} = \begin{pmatrix}U_{1,1}\\ U_{-1,1}\end{pmatrix}$$

Φ	Bosonic form	X_Φ	$Z_2 \times Z_2$	$1 \leftrightarrow 2$	D_1	D_2	$D_\pm$
σ_1		$\frac{1}{8}$	$- \times +$	σ_2	μ_1	σ_1	μ_1
σ_2		$\frac{1}{8}$	$+ \times -$	σ_1	$-\sigma_2$	μ_2	$\pm\mu_2$
μ_1		$\frac{1}{8}$	$+ \times +$	μ_2	σ_1	μ_1	σ_1
μ_2		$\frac{1}{8}$	$+ \times +$	μ_1	μ_2	σ_2	$\mp\sigma_2$
ψ_1^+	$\cos\left(\frac{2\pi}{\beta}\tilde{\varphi} + \frac{\beta}{2}\varphi\right)$	$\frac{\beta^2}{16\pi} + \frac{\pi}{\beta^2}$	$- \times +$	ψ_2^+	$+$	$-$	$-$
ψ_1^-	$\cos\left(\frac{2\pi}{\beta}\tilde{\varphi} - \frac{\beta}{2}\varphi\right)$	$\frac{\beta^2}{16\pi} + \frac{\pi}{\beta^2}$	$- \times +$	ψ_2^-	$+$	$+$	$+$
ψ_2^+	$\sin\left(\frac{2\pi}{\beta}\tilde{\varphi} + \frac{\beta}{2}\varphi\right)$	$\frac{\beta^2}{16\pi} + \frac{\pi}{\beta^2}$	$+ \times -$	ψ_1^+	$-$	$+$	$-$
ψ_2^-	$\sin\left(\frac{2\pi}{\beta}\tilde{\varphi} - \frac{\beta}{2}\varphi\right)$	$\frac{\beta^2}{16\pi} + \frac{\pi}{\beta^2}$	$+ \times -$	ψ_1^-	$+$	$+$	$+$
$\mathcal{E} = \varepsilon_1 + \varepsilon_2$	$\cos\beta\varphi$	$\frac{\beta^2}{4\pi}$	$+ \times +$	$+$	$\mathcal{C}$	$-\mathcal{C}$	$-$
$\mathcal{C} = \varepsilon_1 - \varepsilon_2$	$\cos\frac{4\pi}{\beta}\tilde{\varphi}$	$\frac{4\pi}{\beta^2}$	$+ \times +$	$-$	$\mathcal{E}$	$-\mathcal{E}$	$-$
$\mathcal{E}_+ = \psi_2^+\psi_1^- + \psi_1^+\psi_2^-$	$\sin\frac{4\pi}{\beta}\tilde{\varphi}$	$\frac{4\pi}{\beta^2}$	$- \times -$	$+$	$-\mathcal{E}_-$	$\mathcal{E}_-$	$-$
$\mathcal{E}_- = \psi_2^+\psi_1^- - \psi_1^+\psi_2^-$	$\sin\beta\varphi$	$\frac{\beta^2}{4\pi}$	$- \times -$	$-$	$-\mathcal{E}_+$	$\mathcal{E}_+$	$-$
$\mathcal{P} = \sigma_1\sigma_2$	$\sin\frac{\beta}{2}\varphi$	$\frac{\beta^2}{16\pi}$	$- \times -$	$-$	$-\mu_1\sigma_2$	$\sigma_1\mu_2$	$\pm\mu_1\mu_2$
$\mathcal{P}^* = \mu_1\mu_2$	$\cos\frac{\beta}{2}\varphi$	$\frac{\beta^2}{16\pi}$	$+ \times +$	$+$	$\sigma_1\mu_2$	$\mu_1\sigma_2$	$\mp\sigma_1\sigma_2$
$\sigma_1\mu_2$	$\cos\frac{2\pi}{\beta}\tilde{\varphi}$	$\frac{\pi}{\beta^2}$	$- \times +$	$\mu_1\sigma_2$	$\mu_1\mu_2$	$\sigma_1\sigma_2$	$\mp\mu_1\sigma_2$
$\mu_1\sigma_2$	$\sin\frac{2\pi}{\beta}\tilde{\varphi}$	$\frac{\pi}{\beta^2}$	$+ \times -$	$\sigma_1\mu_2$	$-\sigma_1\sigma_2$	$\mu_1\mu_2$	$\pm\sigma_1\mu_2$
$\mathcal{D}_4$	$\cos 2\beta\varphi$	$\frac{\beta^2}{\pi}$	$+ \times +$	$+$	$\tilde{\mathcal{D}}_4$	$\tilde{\mathcal{D}}_4$	$+$
$\tilde{\mathcal{D}}_4$	$\cos\frac{8\pi}{\beta}\tilde{\varphi}$	$\frac{16\pi}{\beta^2}$	$+ \times +$	$+$	$\mathcal{D}_4$	$\mathcal{D}_4$	$+$
$\varepsilon_1\varepsilon_2$	$(\partial_a\varphi)^2$	2	$+ \times +$	$+$	$\left(\frac{4\pi}{\beta^2}\partial_a\tilde{\varphi}\right)^2$	$\left(\frac{4\pi}{\beta^2}\partial_a\tilde{\varphi}\right)^2$	$+$

- **Symmetry transformations:**
  - Spin reversal $(Z_2 \times Z_2)$:

$$\sigma_1 \to -\sigma_1 \qquad \sigma_2 \to \sigma_2$$

$$\varphi \to -\varphi \qquad \tfrac{2\pi}{\beta}\,\tilde{\varphi} \to \pi - \tfrac{2\pi}{\beta}\,\tilde{\varphi}$$

$$\sigma_1 \to \sigma_1 \qquad \sigma_2 \to -\sigma_2$$

$$\varphi \to -\varphi \qquad \tilde{\varphi} \to -\tilde{\varphi}$$

  - Exchange $(1 \leftrightarrow 2)$:

$$\sigma_1 \quad \to \quad \sigma_2$$

$$\varphi \quad \to \quad -\varphi$$

$$\frac{2\pi}{\beta}\,\tilde{\varphi} \quad \to \quad \frac{\pi}{2} - \frac{2\pi}{\beta}\,\tilde{\varphi}$$

  - Semi-dualities (D_j):

$$\sigma_1 \to \mu_1 \qquad \sigma_2 \to -\sigma_2$$

$$\tfrac{\beta}{2}\,\varphi \leftrightarrow -\tfrac{2\pi}{\beta}\,\tilde{\varphi}$$

$$\sigma_1 \to \sigma_1 \qquad \sigma_2 \to \mu_2$$

$$\frac{\beta}{2}\,\varphi \leftrightarrow \frac{\pi}{2} - \frac{2\pi}{\beta}\,\tilde{\varphi}$$

  - Dualities $(D_\pm)$:

$$\sigma_1 \quad \to \mu_1$$

$$\sigma_2 \quad \to \pm\mu_2$$

$$\beta\varphi \quad \to \quad \beta\varphi \pm \pi$$

$$\frac{2\pi}{\beta}\,\tilde{\varphi} \quad \to \quad \frac{2\pi}{\beta}\,\tilde{\varphi} \pm \frac{\pi}{2}$$

The self-dual line

$$\mathcal{A}_{sd} = \int d^2x \left[\frac{1}{2}(\partial_a \varphi)^2 - \sum_n \left(g_n \cos 2n\beta\varphi + \tilde{g}_n \cos 8n\frac{\pi}{\beta}\tilde{\varphi} \right) \right]$$

– invariant under all the previous transformations;

– g_n, $\tilde{g}_n$, β non-universal functions of J_4;

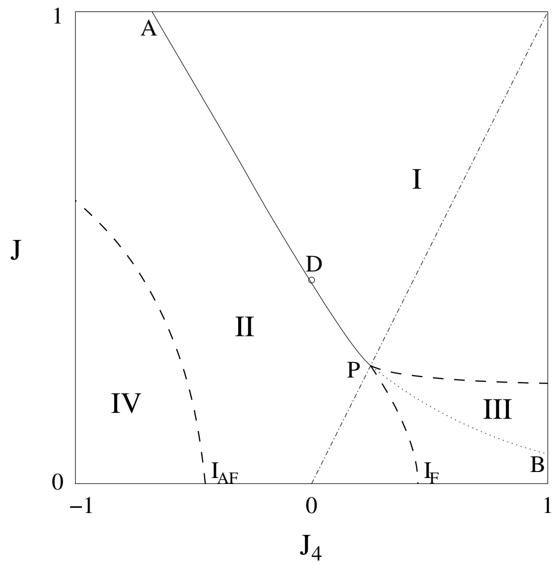
– criticality as long as all perturbing operators are irrelevant :

$\cos 2\beta\varphi$ marginal at $\beta^2 = 2\pi$

$\cos 8\pi\tilde{\varphi}/\beta$ marginal at $\beta^2 = 8\pi$

$\Longrightarrow$ the critical range of $\mathcal{A}_{sd}$ is $\quad 2\pi \le \beta^2 \le 8\pi$

$X_{\sigma_1} = X_{\sigma_2} = X_{\sigma_1\sigma_2}$ at $\quad \beta^2 = 2\pi \quad \Longrightarrow \quad$ Potts point



Comparison with Baxter's solution yields

$$\frac{4\pi}{\beta^2} = 1 - \frac{2}{\pi}\arcsin\left(\frac{\tanh 2J_4}{\tanh 2J_4 - 1}\right)$$

for the square lattice (Kadanoff).

$$\beta^2 \to 6\pi \quad \text{as} \quad J_4 \to -\infty \quad \Longrightarrow$$

the AT critical line corresponds to $\quad 2\pi \le \beta^2 < 6\pi$

Breaking duality

$\cos(2n-1)\beta\varphi$ odd under duality $\quad\Longrightarrow$

$$\mathcal{A}_{AT} = \mathcal{A}_{sd} - \sum_n \tau_n \int d^2x \, \cos(2n-1)\beta\varphi$$

is the action of the scaling AT model;

τ_n non-universal function of J and J_4.

Ignoring irrelevant terms for $2\pi \leq \beta^2 < 6\pi$ yields

$$\mathcal{A}_{SG} = \int d^2x \left(\frac{1}{2} (\partial_a \varphi)^2 - \tau \cos \beta\varphi \right)$$

- $\tau > 0$: disordered phase II

vacuum $|0\rangle$ at $\varphi = 0$ invariant under spin reversal and exchange symmetry

$$\langle 0|\sigma_1|0\rangle = \langle 0|\sigma_2|0\rangle = 0$$
$$\langle 0|\sigma_1\sigma_2|0\rangle = \langle 0| \sin \frac{\beta\varphi}{2} |0\rangle = 0$$

- $\tau < 0$: ordered region I

2 vacua $|0_\pm\rangle = D_\pm|0\rangle$ at $\beta\varphi = \pm\pi$

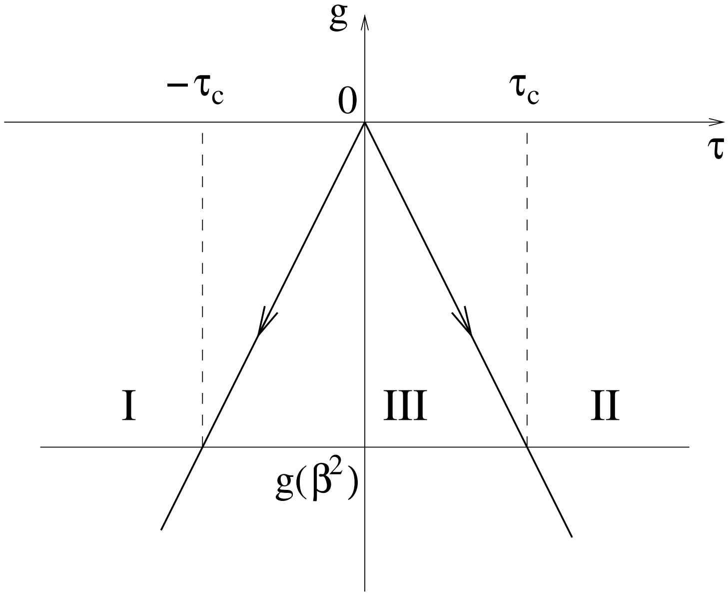
$|0_+\rangle \leftrightarrow |0_-\rangle$ under spin reversal and exchange symmetry $\Rightarrow$ spontaneous symmetry breaking

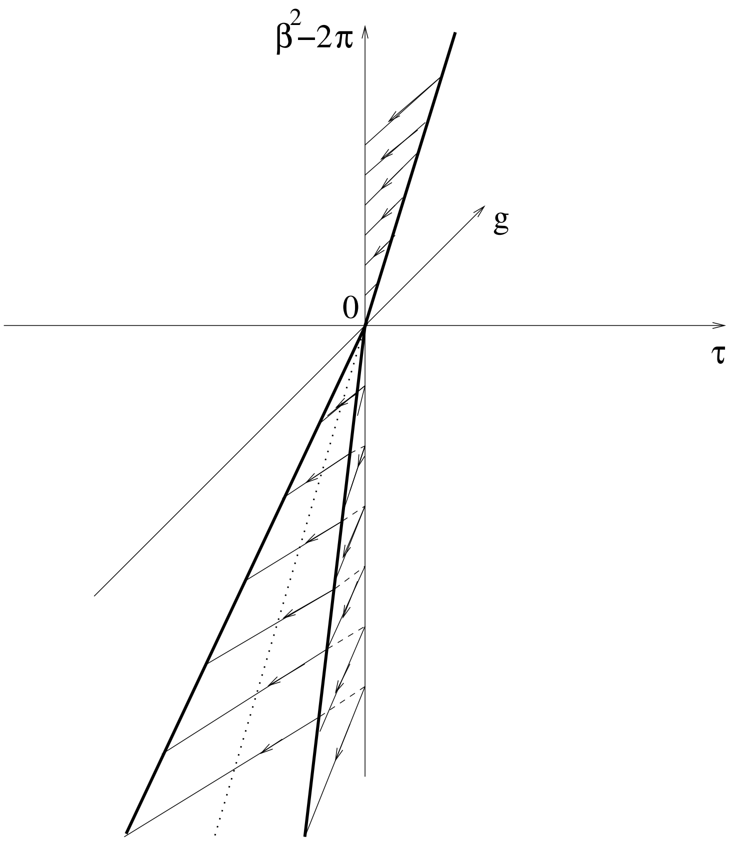
$$\langle 0_+|\sigma_1|0_+\rangle = \langle 0_+|\sigma_2|0_+\rangle = \pm M_\sigma$$
$$\langle 0_-|\sigma_1|0_-\rangle = -\langle 0_-|\sigma_2|0_-\rangle = \pm M_\sigma$$
$$\langle 0_\pm|\sigma_1\sigma_2|0_\pm\rangle = \langle 0_\pm| \sin \frac{\beta\varphi}{2} |0_\pm\rangle = \pm M_{\mathcal{P}}$$

Bifurcation at the Potts point

For $\beta^2 < 2\pi$ more operators become relevant

$$\mathcal{A}_{DSG} = \int d^2x \left(\frac{1}{2} (\partial_a \varphi)^2 - \tau \cos \beta\varphi - g \cos 2\beta\varphi \right)$$





Scattering description of phases I and II

$A_\pm = (A_1 \pm i A_2)/\sqrt{2}$ sine-Gordon soliton/antisoliton

$$A_i(\theta_1)A_j(\theta_2) = \sum_{k,l=1,2} S_{A_iA_j}^{A_kA_l}(\theta_1-\theta_2)A_l(\theta_2)A_k(\theta_1)$$

$$S_{A_1A_1}^{A_1A_1}(\theta) = S_{A_2A_2}^{A_2A_2}(\theta) = \frac{S(\theta)+S_+(\theta)}{2}$$
$$S_{A_1A_1}^{A_2A_2}(\theta) = S_{A_2A_2}^{A_1A_1}(\theta) = \frac{S_+(\theta)-S(\theta)}{2}$$
$$S_{A_1A_2}^{A_1A_2}(\theta) = S_{A_2A_1}^{A_2A_1}(\theta) = \frac{S(\theta)+S_-(\theta)}{2}$$
$$S_{A_1A_2}^{A_2A_1}(\theta) = S_{A_2A_1}^{A_1A_2}(\theta) = \frac{S(\theta)-S_-(\theta)}{2}$$

$$S_+(\theta) = -\frac{\sinh\frac{\pi}{2\xi}(\theta+i\pi)}{\sinh\frac{\pi}{2\xi}(\theta-i\pi)}S(\theta)$$
$$S_-(\theta) = -\frac{\cosh\frac{\pi}{2\xi}(\theta+i\pi)}{\cosh\frac{\pi}{2\xi}(\theta-i\pi)}S(\theta)$$
$$S(\theta) = -\exp\left\{-i\int_0^\infty \frac{dx}{x}\frac{\sinh\frac{x}{2}\left(1-\frac{\xi}{\pi}\right)}{\sinh\frac{x\xi}{2\pi}\cosh\frac{x}{2}}\sin\frac{\theta x}{\pi}\right\}$$

$$\xi = \frac{\pi\beta^2}{8\pi-\beta^2}$$

Breathers: $A_jA_j \to B_{2k}\,, \qquad A_1A_2 \to B_{2k+1}$

$$m_{B_n} = 2M\sin(n\xi/2) \qquad 1 \le n < [\pi/\xi]$$

A_1, A_2 are ordinary particles in phase II and kinks in phase I

Form factors

$$F^{\Phi}_{a_1\dots a_n}(\theta_1,\dots,\theta_n) = \langle 0|\Phi(0)|a_1(\theta_1)\dots a_n(\theta_n)\rangle$$

Results for sine-Gordon: Karowski-Weisz, Smirnov, Lukyanov, Lukyanov-A. Zamolodchikov, ...

Two-particle ff's: $F^{\Phi}_{ab}(\theta_1-\theta_2) = \langle 0|\Phi(0)|a(\theta_1)b(\theta_2)\rangle$

$$F^{\Phi}_{ab}(\theta) = S^{cd}_{ab}(\theta)F^{\Phi}_{dc}(-\theta)$$
$$F^{\Phi}_{ab}(\theta+2i\pi) = l_{\Phi,a}F^{\Phi}_{ba}(-\theta)$$
$$\mathrm{Res}_{\theta=i\pi}F^{\Phi}_{\bar{a}b}(\theta) = i\delta_{ab}\,(1-l_{\Phi,a})\langle\Phi\rangle$$
$$\mathrm{Res}_{\theta=iu^c_{ab}}F^{\Phi}_{ab}(\theta) = i\Gamma^c_{ab}\,F^{\Phi}_c$$

Particle	Creating operator	$Z_2\times Z_2$	$1\leftrightarrow 2$
A_1	$\sigma_1\mu_2$	$-\times+$	A_2
A_2	$\mu_1\sigma_2$	$+\times-$	A_1
B_{2k+1}	$\sin\beta\varphi$	$-\times-$	$-$
B_{2k}	$\cos\beta\varphi$	$+\times+$	$+$

Φ	Bosonic form	l_{Φ,A_j}	l_{Φ,B_n}
σ_i		$(-1)^{\delta_{ij}+1}$	$(-1)^n$
μ_i		$(-1)^{\delta_{ij}}$	$(-1)^n$
$\mathcal{E}=\varepsilon_1+\varepsilon_2$	$\cos\beta\varphi$	$+1$	$+1$
$\mathcal{C}=\varepsilon_1-\varepsilon_2$	$\cos\frac{4\pi}{\beta}\tilde{\varphi}$	$+1$	$+1$
$\mathcal{E}_+=\bar{\psi}_1\psi_2+\bar{\psi}_2\psi_1$	$\sin\frac{4\pi}{\beta}\tilde{\varphi}$	$+1$	$+1$
$\mathcal{E}_-=\bar{\psi}_1\psi_2-\bar{\psi}_2\psi_1$	$\sin\beta\varphi$	$+1$	$+1$
$\mathcal{P}=\sigma_1\sigma_2$	$\sin\frac{\beta}{2}\varphi$	-1	$+1$
$\mathcal{P}^*=\mu_1\mu_2$	$\cos\frac{\beta}{2}\varphi$	-1	$+1$
$\sigma_1\mu_2$	$\cos\frac{2\pi}{\beta}\tilde{\varphi}$	$+1$	$+1$
$\mu_1\sigma_2$	$\sin\frac{2\pi}{\beta}\tilde{\varphi}$	$+1$	$+1$

Asymptotic bound: $\lim_{|\theta|\to\infty}F^{\Phi}_{ab}(\theta)\sim\exp(y_\Phi|\theta|)$

$$y_\Phi \le \frac{X_\Phi}{2}$$

In particular, in the paramagnetic phase II:

- Disorder operators μ_1, μ_2:

$$F^{\mu_l}_{A_jA_j}(\theta) = \frac{i\pi\langle\mu_l\rangle}{2\xi\omega(i\pi)}\frac{F_0(\theta)}{\sinh\frac{\pi}{2\xi}(\theta-i\pi)}\left[\omega(\theta)+(-1)^{l+j}\omega(2i\pi-\theta)\right]$$

$$\omega(\theta) = \exp\left\{2\int_0^\infty \frac{dx}{x}\,\frac{\sinh\left(1-\frac{\xi}{\pi}\right)x}{\sinh\frac{x\xi}{\pi}}\,\frac{\sin^2\frac{\theta x}{2\pi}}{\sinh 2x}\right\}$$

$$F_0(\theta) = -i\sinh\frac{\theta}{2}\,\exp\left\{\int_0^\infty \frac{dx}{x}\,\frac{\sinh\frac{x}{2}\left(1-\frac{\xi}{\pi}\right)}{\sinh\frac{x\xi}{2\pi}\,\cosh\frac{x}{2}}\,\frac{\sin^2\frac{(i\pi-\theta)x}{2\pi}}{\sinh x}\right\}$$

- Order operators σ_1, σ_2:

$$F^{\sigma_{j+1\,(mod\,2)}}_{A_jB_1}(\theta) = -F^{\sigma_{j+1\,(mod\,2)}}_{B_1A_j}(\theta) = (-1)^j\,a\,\frac{T_{(1+\xi/\pi)/2}(\theta)}{\mathcal{P}_{(1+\xi/\pi)/2}(\theta)}$$

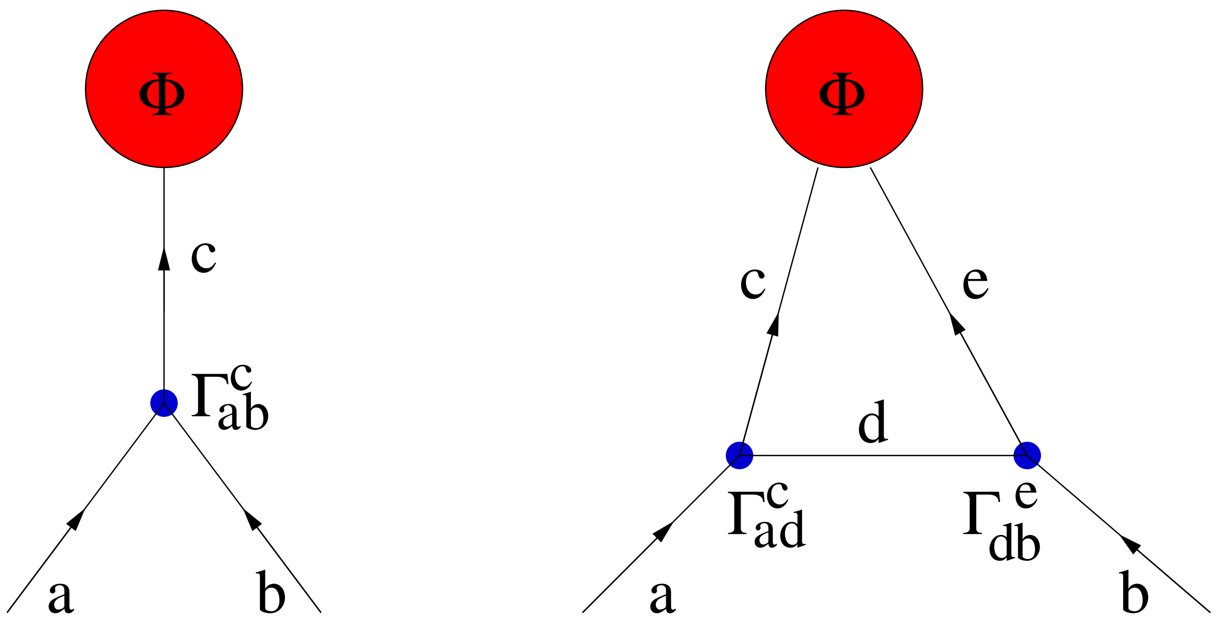
$$F^{\sigma_j}_{A_jB_2}(\theta) = F^{\sigma_j}_{B_2A_j}(\theta) = (b\cosh\theta+c)\,\frac{T_{1/2}(\theta)T_{1/2+\xi/\pi}(\theta)}{\mathcal{P}_{1/2}(\theta)\mathcal{P}_{1/2+\xi/\pi}(\theta)}$$

$$T_\alpha(\theta) = \exp\left\{2\int_0^\infty \frac{dt}{t}\frac{\cosh\left(\alpha-\frac{1}{2}\right)t}{\cosh\frac{t}{2}\sinh t}\sin^2\frac{(i\pi-\theta)t}{2\pi}\right\}$$

$$\mathcal{P}_\alpha(\theta) = \frac{\cos\pi\alpha-\cosh\theta}{2\cos^2\frac{\pi\alpha}{2}}$$

a, b, c determined by

$$\mathrm{Res}_{\theta=i(\pi+\xi)/2}F^{\sigma_2}_{A_1B_1}(\theta) = i\Gamma^{A_2}_{A_1B_1}F^{\sigma_2}_{A_2}$$
$$\mathrm{Res}_{\theta=i(\pi/2+\xi)}F^{\sigma_2}_{A_2B_2}(\theta) = i\Gamma^{A_2}_{A_2B_2}F^{\sigma_2}_{A_2}$$
$$\mathrm{Res}_{\theta=i\pi/2}F^{\sigma_2}_{A_2B_2}(\theta) = i\Gamma^{A_1}_{A_2B_1}\Gamma^{B_1}_{B_1B_2}F^{\sigma_2}_{A_1B_1}(i(\pi-3\xi)/2)$$



Φ
c
Γ_{ab}^{c}
a
b
Φ
c
e
d
Γ_{ad}^{c}
Γ_{db}^{e}
a
b

Correlation functions

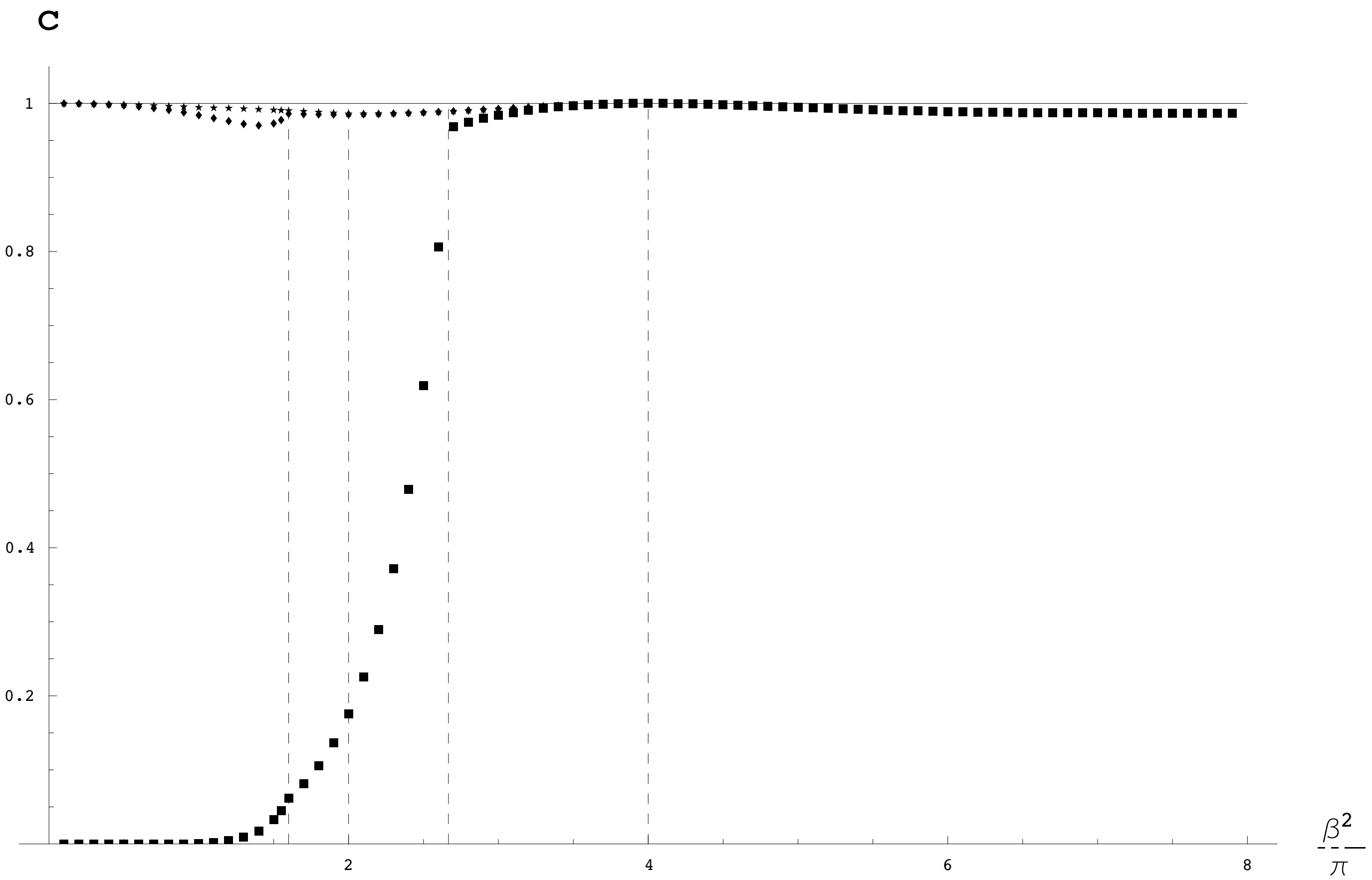
$$\langle \Phi_1(x)\Phi_2(0)\rangle = \sum_{n=0}^{\infty} \sum_{a_1,\ldots,a_n} \int_{\theta_1>\ldots>\theta_n} \frac{d\theta_1}{2\pi}\cdots\frac{d\theta_n}{2\pi}$$

$$\langle 0|\Phi_1(0)|a_1(\theta_1)\ldots a_n(\theta_n)\rangle\langle a_n(\theta_n)\ldots a_1(\theta_1)|\Phi_2(0)|0\rangle e^{-E_n|x|}$$

$$E_n = \sum_{k=1}^{n} m_{a_k}\cosh\theta_k$$

- Example of two-particle approximation: central charge

$$c = \frac{3}{4\pi}\int d^2x\,|x|^2\langle\Theta(x)\Theta(0)\rangle = 1$$



Universal ratios

- Observables : $(\Phi = \sigma_1,\, \sigma_2,\, \sigma_1\sigma_2)$

$M_\Phi = \langle \Phi \rangle$ magnetisations

$C = \int d^2x \, \langle \mathcal{E}(x)\mathcal{E}(0) \rangle$ specific heat

$\chi_\Phi = \int d^2x \, \langle \Phi(x)\Phi(0) \rangle$ susceptibilities

$\chi_{12} = \int d^2x \, \langle \sigma_1(x)\sigma_2(0) \rangle$ mixed susceptibility

$\xi_\Phi^{2nd} = \left(\frac{1}{4\chi_\Phi} \int d^2x \, |x|^2 \langle \Phi(x)\Phi(0) \rangle \right)^{1/2}$ 2nd moment c. l.

$\lim_{|x|\to\infty} \langle \Phi(x)\Phi(0) \rangle \sim \exp(-|x|/\xi_\Phi)$ 'true' c. l.

- For the observable $\mathcal{F}$:

$\mathcal{F}^+$ is the limit toward a critical point along a path in phase II

$\mathcal{F}^-$ is the limit toward a critical point along a dual path in phase I

- Examples of universal ratios :

$$\frac{\chi_\Phi^+}{\chi_\Phi^-}$$

$$R_\Phi = \frac{\chi_\Phi^+}{(\xi_\sigma^+ M_\Phi^-)^2}$$

β^2/π	$\frac{\xi_\sigma^{2nd\,+}}{\xi_\sigma^+}$	$\frac{\xi_\sigma^{2nd\,+}}{\xi_\sigma^{2nd\,-}}$	$\chi_\sigma^+/\chi_\sigma^-$	$\chi_{12}^-/\chi_\sigma^-$	R_σ	$\frac{\xi_\mathcal{P}^{2nd\,+}}{\xi_\sigma^+}$	$\frac{\xi_\mathcal{P}^{2nd\,+}}{\xi_\mathcal{P}^{2nd\,-}}$	$\chi_\mathcal{P}^+/\chi_\mathcal{P}^-$	$R_\mathcal{P}$
2.0	0.9905	1.964	11.59	0.9411	1.434	0.9905	1.964	11.59	1.434
2.1	0.9923	2.070	13.42	0.9212	1.501	0.9322	1.931	10.79	1.320
2.2	0.9938	2.176	15.33	0.8970	1.563	0.8795	1.897	10.06	1.218
2.3	0.9950	2.279	17.32	0.8685	1.619	0.8316	1.863	9.392	1.125
2.4	0.9961	2.381	19.33	0.8353	1.670	0.7880	1.829	8.774	1.040
2.5	0.9970	2.479	21.36	0.7976	1.717	0.7482	1.795	8.202	0.9626
2.6	0.9978	2.573	23.35	0.7555	1.758	0.7117	1.760	7.670	0.8915
8/3	0.9983	2.634	24.64	0.7250	1.784	0.6891	1.737	7.334	0.8473
2.7	0.9984	2.662	25.29	0.7093	1.796	0.6781	1.725	7.175	0.8264
2.8	0.9988	2.745	27.15	0.6595	1.830	0.6470	1.689	6.720	0.7672
2.9	0.9991	2.821	28.89	0.6065	1.861	0.6181	1.654	6.295	0.7127
3.0	0.9993	2.890	30.51	0.5511	1.888	0.5915	1.618	5.897	0.6623
3.1	0.9995	2.950	31.96	0.4939	1.911	0.5668	1.583	5.525	0.6156
3.2	0.9997	3.002	33.26	0.4356	1.931	0.5438	1.548	5.175	0.5724
3.3	0.9998	3.046	34.38	0.3769	1.948	0.5224	1.513	4.847	0.5323
3.4	0.9999	3.081	35.33	0.3184	1.963	0.5026	1.479	4.538	0.4950
3.5	0.9999	3.110	36.10	0.2608	1.975	0.4840	1.445	4.247	0.4602
3.6	1.000	3.131	36.71	0.2045	1.984	0.4668	1.412	3.972	0.4278
3.7	1.000	3.146	37.16	0.1500	1.991	0.4506	1.380	3.711	0.3976
3.8	1.000	3.155	37.47	0.09763	1.996	0.4355	1.349	3.464	0.3694
3.9	1.000	3.161	37.64	0.04756	1.999	0.4214	1.319	3.228	0.3430
4.0	1	3.162	37.70	0	2	0.4082	1.291	3	0.3183
4.1	1	3.161	37.65	0.04494	1.999	0.3959	1.264	2.782	0.2952
4.2	1	3.157	37.50	0.08721	1.996	0.3843	1.238	2.579	0.2736
4.3	1	3.152	37.28	0.1268	1.992	0.3735	1.213	2.389	0.2533
4.4	1	3.145	36.98	0.1637	1.986	0.3634	1.189	2.211	0.2343
4.5	1	3.138	36.63	0.1981	1.979	0.3539	1.165	2.044	0.2165
4.6	1	3.130	36.23	0.2299	1.971	0.3450	1.142	1.887	0.1998
4.7	1	3.122	35.78	0.2594	1.961	0.3366	1.120	1.740	0.1841
4.8	1	3.114	35.30	0.2866	1.950	0.3288	1.099	1.603	0.1694
4.9	1	3.105	34.80	0.3117	1.937	0.3215	1.078	1.473	0.1556
5.0	1	3.097	34.28	0.3348	1.924	0.3147	1.059	1.352	0.1427
5.1	1	3.090	33.74	0.3560	1.909	0.3082	1.040	1.238	0.1306
5.2	1	3.082	33.18	0.3754	1.893	0.3022	1.022	1.132	0.1193
5.3	1	3.075	32.62	0.3932	1.877	0.2966	1.005	1.032	0.1087
5.4	1	3.069	32.05	0.4095	1.859	0.2914	0.9894	0.9383	0.09876
5.5	1	3.063	31.48	0.4244	1.840	0.2865	0.9742	0.8509	0.08950
5.6	1	3.057	30.91	0.4380	1.821	0.2819	0.9599	0.7691	0.08086
5.7	1	3.052	30.33	0.4503	1.800	0.2776	0.9463	0.6929	0.07280
5.8	1	3.047	29.76	0.4616	1.778	0.2736	0.9336	0.6218	0.06530
5.9	1	3.043	29.18	0.4719	1.756	0.2699	0.9216	0.5557	0.05834
6.0	1	3.039	28.60	0.4812	1.732	0.2664	0.9104	0.4944	0.05188

- Results at the Ising decoupling point $\beta^2 = 4\pi$:

Ratio	Two-particle approximation	Exact
$\xi_\sigma^{2nd+}/\xi_\sigma^+$	1	0.999598087..
$\xi_\sigma^{2nd+}/\xi_\sigma^{2nd-}$	3.1623	3.16249504..
$\chi_\sigma^+/\chi_\sigma^-$	37.699	37.6936520..
R_σ	2	2.00163051..
$\xi_{\mathcal{P}}^{2nd+}/\xi_\sigma^+$	0.408	0.40656..
$\xi_{\mathcal{P}}^{2nd+}/\xi_{\mathcal{P}}^{2nd-}$	1.291	1.3088..
$\chi_{\mathcal{P}}^+/\chi_{\mathcal{P}}^-$	3	2.9108..
$R_{\mathcal{P}}$	0.318	0.32104..

$\mathcal{P} = \sigma_1\sigma_2$

• Check at the 4-state Potts point $\beta^2 = 2\pi$:

$\sigma = (\sigma_1, \sigma_2) \qquad\qquad \sigma_1,\ \sigma_2 = \pm 1$

Potts spin variables :

$$\begin{aligned} s_\alpha(x) &= \delta_{\sigma(x),\alpha} - \frac{1}{4} \\ &= \frac{1}{4}[\alpha_1\sigma_1(x) + \alpha_2\sigma_2(x) + \alpha_1\alpha_2\sigma_1\sigma_2(x)] \end{aligned}$$

Potts susceptibilities :

$$\chi = \int d^2x \, \langle s_\alpha(x) s_\alpha(0) \rangle \qquad\qquad T > T_c$$

$$\chi_L = \int d^2x \, \langle 0_\alpha | s_\alpha(x) s_\alpha(0) | 0_\alpha \rangle \qquad\qquad T < T_c$$

$$\chi_T = \int d^2x \, \langle 0_\alpha | s_\gamma(x) s_\gamma(0) | 0_\alpha \rangle \qquad \alpha \neq \gamma \qquad\qquad T < T_c$$

$$\chi^+/\chi_L^- = \left[\frac{\chi_\sigma^+/\chi_\sigma^-}{1 + 2\chi_{12}^-/\chi_\sigma^-} \right]_{\beta^2 = 2\pi} \simeq 4.02$$

$$\chi_T^-/\chi_L^- = \left[\frac{1 - \frac{2}{3}\chi_{12}^-/\chi_\sigma^-}{1 + 2\chi_{12}^-/\chi_\sigma^-} \right]_{\beta^2 = 2\pi} \simeq 0.129$$

in agreement with G.D.-J. Cardy (1998, 2000)

Lattice estimates :

– Caselle, Tateo and Vinti (1999), numerical simulations

– Enting and Guttmann (2002), series expansions

compatible with field theory predictions but large error bars (logarithms)