Lattice Supersymmetry and Supersymmetry breaking: the Wess-Zumino model

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Outline

- Exact supersymmetry on the lattice
  - $d = 4$ $N = 1$ Wess-Zumino model using the Ginsparg-Wilson relation

M. Bonini and A. F.
JHEP 09 (2004) 011 [hep-lat/0402034]

- Dynamical supersymmetry breaking
  - $d = 2$ $N = 1$ Wess-Zumino model in the Hamiltonian formulation

M. Beccaria, M. Campostrini and A. F.
Phys. Rev. D69 (2004) 095010 [hep-lat/0402007]
M. Beccaria, G. F. De Angelis, M. Campostrini and A. F.
Phys. Rev. D70 (2004) 035011 [hep-lat/0405016]

- Conclusions and Outlook

Exact supersymmetry on the lattice

Generalized lattice supersymmetry transformation that leaves the 4 dimensional $N = 1$ Wess-Zumino invariant

The major obstacle in formulating a supersymmetric theory on the lattice arises from the fact that the supersymmetry algebra is actually an extension of the Poincaré algebra, which is explicitly broken by the lattice.

Indeed, in an interacting theory, translation invariance is broken since the Leibniz rule is not valid for lattice derivatives [Dondi 77].

We know that Poincaré invariance is achieved automatically in the continuum limit without fine tuning since operators that violate Poincaré invariance are all irrelevant.

A related issue is how to balance bosonic and fermionic degrees of freedom, as demanded by supersymmetry: a naive regularization of fermions produces a wrong number of them. In order to construct a chiral lattice theory, without fermion doubling, the lattice Dirac operator D should satisfy the Ginsparg-Wilson relation [Ginsparg-Wilson 82].

It is possible to obtain exact supersymmetry respect to the supersymmetric transformations. An incomplete list is ...

Golterman & Petcher, Nucl.Phys.B319:307-341,1989
Bietenholz, Mod.Phys.Lett.A14:51-62,1999
Catterall & Karamov, Phys.Rev.D65:094501,2002,
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Four dimensional lattice Wess-Zumino model with GW fermions

Our starting point is the paper by Fujikawa (2002)

We show that it is actually possible to formulate the theory in such a way that the full action is invariant under a lattice superymmetry transformation at fixed lattice spacing.

The action and the transformation are written in terms of the Ginsparg-Wilson operator and reduce to their continuum expression in the limit $a \to 0$.

The lattice supersymmetry transformation is non-linear in the scalar fields and depends on the parameters m and g entering in the superpotential.

We also show that the lattice supersymmetry transformation close the algebra, a necessary ingredient to guarantee the request of supersymmetry.

The Ginsparg-Wilson relation

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

implies a continuum symmetry of the fermion action which may be regarded as a lattice form of the chiral symmetry [Lüscher 1998].

The fermion lagrangian with a Yukawa interaction

$$\mathcal{L} = \bar{\psi} D\psi + g\bar{\psi}(P_+\phi\hat{P}_+ + P_-\phi^\dagger\hat{P}_-)\psi\,,$$

where

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5)\,, \qquad \hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5)$$

are the lattice chiral projection operators and $\hat{\gamma}_5 = \gamma_5(1 - aD)$, is invariant under the lattice chiral transformation

$$\delta\psi = i\varepsilon\hat{\gamma}_5\psi\,, \qquad \delta\bar{\psi} = i\bar{\psi}\gamma_5\varepsilon\,, \qquad \delta\phi = -2i\varepsilon\phi\,.$$

By writing ψ in terms of two Majorana fermions

$$\psi = \chi + i\eta \,,$$

it can be seen that the interaction term couples the two Majorana fermions and therefore there is a conflict between lattice chiral symmetry and the Majorana condition [Fujikawa 2002].

This is due to the fact that the projection operators $\hat{P}_{\pm}$ depend on D. By making the following field redefinition

$$\psi' = (1 - \frac{a}{2}D)\psi \,, \qquad \bar{\psi}' = \bar{\psi} \,,$$

the Yukawa interaction becomes

$$g\bar{\psi}'(P_+ \phi P_+ + P_- \phi^\dagger P_-)\psi'$$

and the two Majorana components of ψ' decouple.

Taking advantage of this property, one can define the four dimensional Wess-Zumino on the lattice with Majorana fermions.

We start with a lagrangian defined in terms of the Ginsparg-Wilson fermions on the $d = 4$ euclidean lattice. A simple solution was given by Neuberger [Neuberger 1998]

$$D = \frac{1}{a}\big(1 - \frac{X}{\sqrt{X^\dagger X}}\big)\,, \qquad\qquad X = 1 - aD_w\,,$$

where

$$D_w = \frac{1}{2}\gamma_\mu(\nabla^\star_\mu + \nabla_\mu) - \frac{a}{2}\nabla^\star_\mu\nabla_\mu$$

and

$$\nabla_\mu\phi(x) = \frac{1}{a}(\phi(x + a\widehat{\mu}) - \phi(x))\,, \qquad \nabla^\star_\mu\phi(x) = \frac{1}{a}(\phi(x) - \phi(x - a\widehat{\mu}))$$

It is convenient to write

$$D = D_1 + D_2$$

where

$$D_1 = \frac{1}{a}\Big(1 - \frac{1 + \frac{a^2}{2}\nabla^\star_\mu \nabla_\mu}{\sqrt{X^\dagger X}}\Big)\,, \qquad D_2 = \frac{1}{2}\gamma_\mu \frac{\nabla^\star_\mu + \nabla_\mu}{\sqrt{X^\dagger X}} \equiv \gamma_\mu D_{2\mu}\,.$$

In terms of D_1 and D_2 the Ginsparg-Wilson relation becomes

$$D_1^2 - D_2^2 = \frac{2}{a} D_1\,.$$

The action of the 4-dimensional Wess-Zumino model on the lattice

$$
\begin{aligned}
S_{WZ} &= \sum_x \{\frac{1}{2}\bar{\chi}(1-\frac{a}{2}D_1)^{-1}D_2\chi - \frac{2}{a}\phi^\dagger D_1\phi + F^\dagger(1-\frac{a}{2}D_1)^{-1}F + \frac{1}{2}m\bar{\chi}\chi \\
&\quad +m(F\phi + (F\phi)^\dagger) + g\bar{\chi}(P_+\phi P_+ + P_-\phi^\dagger P_-)\chi + g(F\phi^2 + (F\phi^2)^\dagger)\}\,,
\end{aligned}
$$

where ϕ and F are scalar fields and χ is a Majorana fermion which satisfies the Majorana condition

$$\bar{\chi} = \chi^T C$$

and C is the charge conjugation matrix which satisfies

$$C^T = -C\,, \qquad CC^\dagger = 1\,.$$

Moreover, our conventions are

$$
\begin{aligned}
C\gamma_\mu C^{-1} &= -(\gamma_\mu)^T \\
C\gamma_5 C^{-1} &= (\gamma_5)^T\,.
\end{aligned}
$$

In the continuum limit reduces to the continuum Wess-Zumino action

$$
\begin{aligned}
S \;=\; & \int \Big\{ \frac{1}{2}\bar{\chi}(\not{\partial} + m)\chi + \phi^{\dagger}\partial^{2}\phi + F^{\dagger}F + m(F\phi + (F\phi)^{\dagger}) \\
& + g\bar{\chi}(P_{+}\phi P_{+} + P_{-}\phi^{\dagger}P_{-})\chi + g(F\phi^{2} + (F\phi^{2})^{\dagger}) \Big\} \,.
\end{aligned}
$$

The supersymmetric transformation

If one defines the real components by

$$\phi \to \frac{1}{\sqrt{2}}(A + iB)\,, \quad F \to \frac{1}{\sqrt{2}}(F - iG)$$

the WZ model $S_{WZ} = S_0 + S_{int}$

$$S_0 = \sum_x \{\frac{1}{2}\bar{\chi}(1 - \frac{a}{2}D_1)^{-1} D_2\chi - \frac{1}{a}(AD_1A + BD_1B)$$
$$+ \frac{1}{2}F(1 - \frac{a}{2}D_1)^{-1}F + \frac{1}{2}G(1 - \frac{a}{2}D_1)^{-1}G\}\,,$$

$$S_{int} = \sum_x \{\frac{1}{2}m\bar{\chi}\chi + m(FA + GB) + \frac{1}{\sqrt{2}}g\bar{\chi}(A + i\gamma_5 B)\chi$$
$$+ \frac{1}{\sqrt{2}}g[F(A^2 - B^2) + 2G(AB)]\}\,.$$

S_0 (free part) is invariant under the lattice supersymmetry transformation [Fujikawa 2002]

$$
\begin{aligned}
&\delta A = \bar{\varepsilon}\chi = \bar{\chi}\varepsilon \\
&\delta B = -i\bar{\varepsilon}\gamma_5\chi = -i\bar{\chi}\gamma_5\varepsilon \\
&\delta\chi = -D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon \\
&\delta F = \bar{\varepsilon}D_2\chi \\
&\delta G = i\bar{\varepsilon}D_2\gamma_5\chi\,.
\end{aligned}
$$

In fact, the variation of S_0 under the this transformation is

$$
\begin{aligned}
&\delta S_0 = \\
&= \sum_x \{\bar{\chi}(1 - \frac{a}{2}D_1)^{-1}D_2[-D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon] - \frac{2}{a}\bar{\chi}\varepsilon D_1 A \\
&\qquad + \frac{2i}{a}\bar{\chi}\gamma_5\varepsilon D_1 B + (\bar{\varepsilon}D_2\chi)(1 - \frac{a}{2}D_1)^{-1}F + i(\bar{\varepsilon}D_2\gamma_5\chi)(1 - \frac{a}{2}D_1)^{-1}G\}\,.
\end{aligned}
$$

and integrating by part *, this variation becomes

$$\sum_x \{ -\bar{\chi}\varepsilon\big[(1-\frac{a}{2}D_1)^{-1}D_2^2+\frac{2}{a}D_1\big]A + i\bar{\chi}\gamma_5\varepsilon\big[(1-\frac{a}{2}D_1)^{-1}D_2^2+\frac{2}{a}D_1\big]B$$
$$-\bar{\chi}(1-\frac{a}{2}D_1)^{-1}D_2(F-i\gamma_5 G)\varepsilon + \bar{\chi}D_2\varepsilon(1-\frac{a}{2}D_1)^{-1}F + i\bar{\chi}D_2\gamma_5\varepsilon(1-\frac{a}{2}D_1)^{-1}G\}$$
$$= 0\,,$$

where we used the Ginsparg-Wilson relation, which implies

$$(1-\frac{a}{2}D_1)^{-1}D_2^2 = -\frac{2}{a}D_1\,.$$

*For instance, for any scalar function $\mathcal{F}$ one has $\mathcal{F}\bar{\varepsilon}D_2\chi = \bar{\chi}D_2\mathcal{F}\varepsilon$.

Failure of the Leibniz rule

The variation of S_{int} under the susy transformation does not vanish because of the failure of the Leibniz rule at finite lattice spacing [Dondi and Nicolai 1977, Fujikawa 2002]

$$\begin{aligned}
&\frac{1}{a}(f(x+a)g(x+a) - f(x)g(x)) = \\
&= \frac{1}{a}(f(x+a) - f(x))g(x) + \frac{1}{a}f(x)(g(x+a) - g(x)) \\
&+ a\frac{1}{a}(f(x+a) - f(x))\frac{1}{a}(g(x+a) - g(x)) \\
&= (\nabla f(x))g(x) + f(x)(\nabla g(x)) + a(\nabla f(x))(\nabla g(x))
\end{aligned}$$

breaking of supersymmetry is of order $O(a)$.

- In order to discuss the symmetry properties of the lattice Wess-Zumino model one possibility is to modify the action by adding irrelevant terms which make invariant the full action.

- Alternatively, one can modify the supersymmetry transformation in such a way that the action has an exact symmetry for $a \neq 0$.

Since the transformation leaves invariant the free part of the action, this modification must vanish for $g = 0$.

$$\begin{aligned}
&\delta A = \bar{\varepsilon}\chi = \bar{\chi}\varepsilon \\
&\delta B = -i\bar{\varepsilon}\gamma_5\chi = -i\bar{\chi}\gamma_5\varepsilon \\
&\delta\chi = -D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon + gR\varepsilon \\
&\delta F = \bar{\varepsilon}D_2\chi \\
&\delta G = i\bar{\varepsilon}D_2\gamma_5\chi
\end{aligned}$$

- R to be determined by requiring that the variation of the action vanishes.
  - We assume that R depends on the scalar and auxiliary fields and their derivatives and not on χ.

Exact susy transformation for the full action

The variation of the Wess-Zumino action under the transformation is

$$\begin{aligned}
\delta S_{WZ} &= \sum_x \{g\bar{\chi}(1-\frac{a}{2}D_1)^{-1}D_2R\varepsilon - m\bar{\chi}[D_2(A-i\gamma_5 B)\varepsilon + (F-i\gamma_5 G)\varepsilon - gR\varepsilon] \\
&+m(A\bar{\varepsilon}D_2\chi + F\bar{\chi}\varepsilon + iB\bar{\varepsilon}D_2\gamma_5\chi - iG\bar{\chi}\gamma_5\varepsilon) + \frac{g}{\sqrt{2}}\bar{\chi}(\bar{\varepsilon}\chi + \gamma_5(\bar{\varepsilon}\gamma_5\chi))\chi \\
&-\sqrt{2}g\bar{\chi}(A+i\gamma_5 B)[D_2(A-i\gamma_5 B)\varepsilon + (F-i\gamma_5 G)\varepsilon - gR\varepsilon] \\
&+\frac{g}{\sqrt{2}}[(A^2-B^2)\bar{\varepsilon}D_2\chi + 2FA\bar{\chi}\varepsilon + 2iFB\bar{\chi}\gamma_5\varepsilon \\
&+2iAB\bar{\varepsilon}D_2\gamma_5\chi + 2GB\bar{\chi}\varepsilon - 2iGA(\bar{\chi}\gamma_5\varepsilon)]\}\,.
\end{aligned}$$

By using the Fierz identity, terms with four fermions cancel as in the continuum.

Moreover, g independent terms cancel out after an integration by part, and one is left with

$$\begin{aligned}
\delta S_{WZ} &= \sum_x \{g\bar{\chi}[(1-\frac{a}{2}D_1)^{-1}D_2R + mR]\varepsilon - \frac{g}{\sqrt{2}}[2\bar{\chi}(A+i\gamma_5 B)D_2(A-i\gamma_5 B)\varepsilon \\
&\quad - \bar{\chi}D_2(A-i\gamma_5 B)^2\varepsilon] + \sqrt{2}g^2\bar{\chi}(A+i\gamma_5 B)R\varepsilon\}\,.
\end{aligned}$$

The function R is determined by imposing the vanishing of δS_{WZ} order by order in g. By expanding R in powers of g

$$R = R^{(1)} + gR^{(2)} + \cdots$$

and imposing the symmetry condition order by order in perturbation theory, we find

$$R^{(1)} = ((1 - \frac{a}{2}D_1)^{-1}D_2 + m)^{-1}\Delta L$$

with

$$\begin{aligned}\Delta L &\equiv \frac{1}{\sqrt{2}}(2(A + i\gamma_5 B)D_2(A - i\gamma_5 B) - D_2(A - i\gamma_5 B)^2)\\ &= \frac{1}{\sqrt{2}}\{2(AD_2A - BD_2B) - D_2(A^2 - B^2)\\ &\qquad + 2i\gamma_5[(AD_2B + BD_2A) - D_2(AB)]\}\,.\end{aligned}$$

To order g^2 one has

$$R^{(2)} = -\sqrt{2}((1 - \frac{a}{2}D_1)^{-1}D_2 + m)^{-1}(A + i\gamma_5 B)((1 - \frac{a}{2}D_1)^{-1}D_2 + m)^{-1}\Delta L\,,$$

and for $n \geq 2$

$$R^{(n)} = -\sqrt{2}((1 - \frac{a}{2}D_1)^{-1}D_2 + m)^{-1}(A + i\gamma_5 B)R^{(n-1)}\,.$$

The formal solution is

$$[(1 - \frac{a}{2}D_1)^{-1}D_2 + m + \sqrt{2}g(A + i\gamma_5 B)]R = \Delta L\,.$$

- $R \to 0$ for $a \to 0$, since ΔL vanishes in this limit.
- ΔL is different from zero because of the breaking of the Leibniz rule for a finite lattice spacing.

The algebra

By the commutator of two supersymmetries one find a transformation which is still a symmetry of the Wess-Zumino action, i.e. the transformations of the fields form a closed algebra, order by order in g.

Up to order g^1, (the rest can be generalized!)

Two supersymmetry transformations on the scalar field A give

$$\begin{aligned}\delta_1\delta_2 A &= \delta_1(\bar{\varepsilon}_2\chi)\\ &= -\bar{\varepsilon}_2[D_2(A-i\gamma_5 B)\varepsilon_1 + (F-i\gamma_5 G)\varepsilon_1 - gR\varepsilon_1]\end{aligned}$$

and their commutator yields

$$[\delta_2,\delta_1]A = -2\bar{\varepsilon}_1 D_2\varepsilon_2 A + g(\bar{\varepsilon}_1 R\varepsilon_2 - \bar{\varepsilon}_2 R\varepsilon_1)\,.$$

The order g^1 of the second term on the r.h.s. reads

$$\begin{aligned}&g(\bar{\varepsilon}_1 R^{(1)}\varepsilon_2 - \bar{\varepsilon}_2 R^{(1)}\varepsilon_1) =\\ &\sqrt{2}g\bar{\varepsilon}_2\frac{m(1-\frac{a}{2}D_1)}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_2(A^2-B^2)-2(AD_2A-BD_2B)]\varepsilon_1\end{aligned}$$

Finally, the commutator of two supersymmetries on the scalar field A is

$$
\begin{aligned}
[\delta_2,\delta_1]A = & -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}A \\
& + \frac{g}{\sqrt{2}}\frac{m(1-\frac{a}{2}D_1)}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(A^2-B^2)-2(AD_{2\mu}A-BD_{2\mu}B)]\}\,.
\end{aligned}
$$

Similarly, the commutators of two supersymmetries on the other fields, up to terms of order g^1, are

$$
\begin{aligned}
[\delta_2,\delta_1]B = & -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}B \\
& + \sqrt{2}g\frac{m(1-\frac{a}{2}D_1)}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(AB)-(AD_{2\mu}B+BD_{2\mu}A)]\}\,, \\
[\delta_2,\delta_1]F = & -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}F \\
& - \frac{g}{\sqrt{2}}\frac{D_2^2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(A^2-B^2)-2(AD_{2\mu}A-BD_{2\mu}B)]\}\,, \\
[\delta_2,\delta_1]G = & -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}G \\
& - \sqrt{2}g\frac{D_2^2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(AB)-(AD_{2\mu}B+BD_{2\mu}A)]\}
\end{aligned}
$$

and

$$
\begin{aligned}
[\delta_2,\delta_1]\chi = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{ & D_{2\mu}\chi \\
& - \frac{g}{\sqrt{2}}\frac{m(1-\frac{a}{2}D_1)-D_2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}\Big(D_2(A-i\gamma_5 B)\gamma_\mu\chi + (A+i\gamma_5 B)D_2\gamma_\mu\chi \\
& - D_2[(A-i\gamma_5 B)\gamma_\mu\chi]\Big)\Big\}\,.
\end{aligned}
$$

Therefore, the general expression of these commutators is

$$[\delta_1, \delta_2]\Phi = \alpha^\mu P_\mu^\Phi(\Phi)\,, \qquad\qquad \Phi = (A, B, F, G, \chi)\,,$$

where $\alpha^\mu = -2\bar{\varepsilon}_2\gamma^\mu\varepsilon_2$ and $P_\mu^\Phi(\Phi)$ are polynomials in Φ defined as

$$P_\mu^\Phi(\Phi) = D_{2\mu}\Phi + O(g)$$

We have verified that the closure works, i.e. the action is invariant under the transformation (up to terms of order g^1).

$$\Phi \to \Phi + \alpha^\mu P_\mu^\Phi(\Phi)$$

Notice that, in the continuum limit $D_{2\mu} \to \partial_\mu$ and the transformation reduces to

$$\Phi \to \Phi + \alpha^\mu \partial_\mu \Phi$$

Ward-Takahashi identity

We calculated the order $O(g)$ of a simple Ward-Takahashi identity (WTi) and verified that it is satisfied.

Let us start writing down the free propagators of the theory

$$
\begin{aligned}
< AA >^{(0)} &= < BB >^{(0)} = -\mathcal{M}^{-1}(1-\frac{a}{2}D_1)^{-1} \\
< FF >^{(0)} &= < GG >^{(0)} = \frac{2}{a}\mathcal{M}^{-1}D_1 \\
< AF >^{(0)} &= < BG >^{(0)} = m\,\mathcal{M}^{-1} \\
< \chi\bar{\chi} >^{(0)} &= -\mathcal{M}^{-1}((1-\frac{a}{2}D_1)^{-1}D_2 - m)
\end{aligned}
$$

where

$$\mathcal{M} = \frac{2}{a}D_1(1-\frac{a}{2}D_1)^{-1} + m^2.$$

The order $O(g)$ of a simple WTi is the following

$$< -D_2(A - i\gamma_5 B) >^{(1)} - < F >^{(1)} + i\gamma_5 < G >^{(1)} + g < R >^{(0)} = 0.$$

The first term of this WTi is zero because of $\delta-$momentum conservation and $D_2(k = 0) = 0$.

The term $< F_x >^{(1)}$ (a factor $g/\sqrt{2}$ is omitted)

$$< F_x >^{(1)} = < F_x \sum_y (\bar{\chi} A \chi + F A^2 + 2GAB) >^{(0)}$$

in momentum space reads

$$\begin{aligned} < F(k) >^{(1)} = \int_{pq} \{ & < \bar{\chi}(p)\chi(q) >^{(0)} < F(k)A(-p-q) >^{(0)} \\ & + < F(k)F(-p-q) >^{(0)} < A(p)A(q) >^{(0)} \\ & + 2 < F(k)A(-p-q) >^{(0)} < F(p)A(q) >^{(0)} \\ & + 2 < F(k)A(-p-q) >^{(0)} < B(p)G(q) >^{(0)} \}. \end{aligned}$$

By substituting the propagators and using the fact that the F propagator vanishes at zero momentum we have

$$< F(k) >^{(1)} = \frac{\delta^4(k)}{m} \int_p \mathcal{M}^{-1}(p) Tr((1 - \frac{a}{2} D_1(p))^{-1} D_2(p) - m) + 4\delta^4(k) \int_p \mathcal{M}^{-1}(p) = 0.$$

The one point function of the G is zero at the order g.

Finally, the last term of the WTi is

$$< R >^{(0)} = ((1 - \frac{a}{2} D_1)^{-1} D_2 + m)^{-1} < \Delta L >^{(0)}$$

where $\Delta L = 2(AD_2A - BD_2B) - D_2(A^2 - B^2) + 2i\gamma_5[(AD_2B + BD_2A) - D_2(AB)]$. In the following we consider only the contribution from the field A since B can be treated similarly. In momentum space we have

$$< R(k) >^{(0)} = ((1 - \frac{a}{2} D_1(k))^{-1} D_2(k) + m)^{-1} \delta^4(k) \times \{ -2 \int_q D_2(q) \mathcal{M}^{-1}(q) (1 - \frac{a}{2} D_1(q))^{-1} + D_2(k) \int_q \mathcal{M}^{-1}(q) (1 - \frac{a}{2} D_1(q))^{-1} \} = 0.$$

Indeed, the first integrand is an odd function of q thus the first integral vanishes while the second term is zero since $D_2(k = 0) = 0$.

Let us notice that, even if each term of the WTi is zero separately, there are non-trivial cancellations between the fermionic and scalar terms in $< F_x >^{(1)}$.

Dynamical supersymmetry breaking

Study of dynamical supersymmetry breaking by numerical simulations in a class of two dimensional $N=1$ Wess-Zumino model

General remarks

Let us remind the (continuum) $N = 1$ supersymmetry algebra

$$\{Q_\alpha, Q_\beta\} = 2(\not{P}C)_{\alpha\beta}$$

since P_i are not conserved on the lattice, a lattice formulation of a supersymmetric model must break this eq. explicitly.

A very important advantage of the Hamiltonian formulation is the possibility of conserving exactly a key subalgebra of this eq.; specializing to $1 + 1$ dimensions, in a Majorana basis $\gamma_0 = C = \sigma_2$, $\gamma_1 = i\sigma_3$, the eq. becomes

$$Q_1^2 = Q_2^2 = P^0 \equiv H, \qquad \{Q_1, Q_2\} = 2P^1 \equiv 2P,$$

On the lattice, since H is conserved but P is not, we can pick up one of the supercharges, say, $Q_1^2 = H$, build a discretized version Q_L and define the lattice Hamiltonian $H = Q_L^2$.

The explicit lattice model is built by considering a spatial lattice with L sites.

On each site we place a real scalar field φ_n together with its conjugate momentum p_n such that $[p_n, \varphi_m] = -i\delta_{n,m}$. The associated fermion is a Majorana fermion $\psi_{a,n}$ with $a = 1, 2$ and $\{\psi_{a,n}, \psi_{b,m}\} = \delta_{a,b}\delta_{n,m}$, $\psi^\dagger_{a,n} = \psi_{a,n}$.

The discretized supercharge

$$Q_L = \sum_{n=1}^{L} \left[p_n \psi_{1,n} - \left(\frac{\varphi_{n+1} - \varphi_{n-1}}{2} + V(\varphi_n) \right) \psi_{2,n} \right].$$

with arbitrary $V(\varphi)$ (called prepotential) can be used to define a semipositive definite lattice Hamiltonian $H = Q_L^2$

Notice that $Q_1^2 = H$ is enough to guarantee that $E_0 \equiv \langle \Psi_0 | H | \Psi_0 \rangle \geq 0$, that all eigenstates of H with $E > 0$ are paired in doublets and that $E_0 = 0$ if and only if supersymmetry is unbroken, i.e., the ground state is annihilated by Q_1.

Prediction/Obtained results

- All the results for the model with cubic prepotential, $V = \varphi^3$, indicated unbroken supersymmetry.

- Dynamical supersymmetry breaking in the model with quadratic prepotential

$$V = \lambda_2 \varphi^2 + \lambda_0$$

was studied performing numerical simulations along a line of constant λ_2 and confirmed the existence of two phases:

  - a phase of broken SUSY with unbroken discrete Z_2 at high λ_0 and
  - a phase of unbroken SUSY with broken Z_2 at low λ_0, separated by a single phase transition.

What is the value of λ_0 ?

Strong coupling expansion

For polynomial $V(\varphi)$, the relevant parameter is just its degree q.

- For odd q strong coupling expansion and tree-level results agree and supersymmetry is expected to be unbroken.

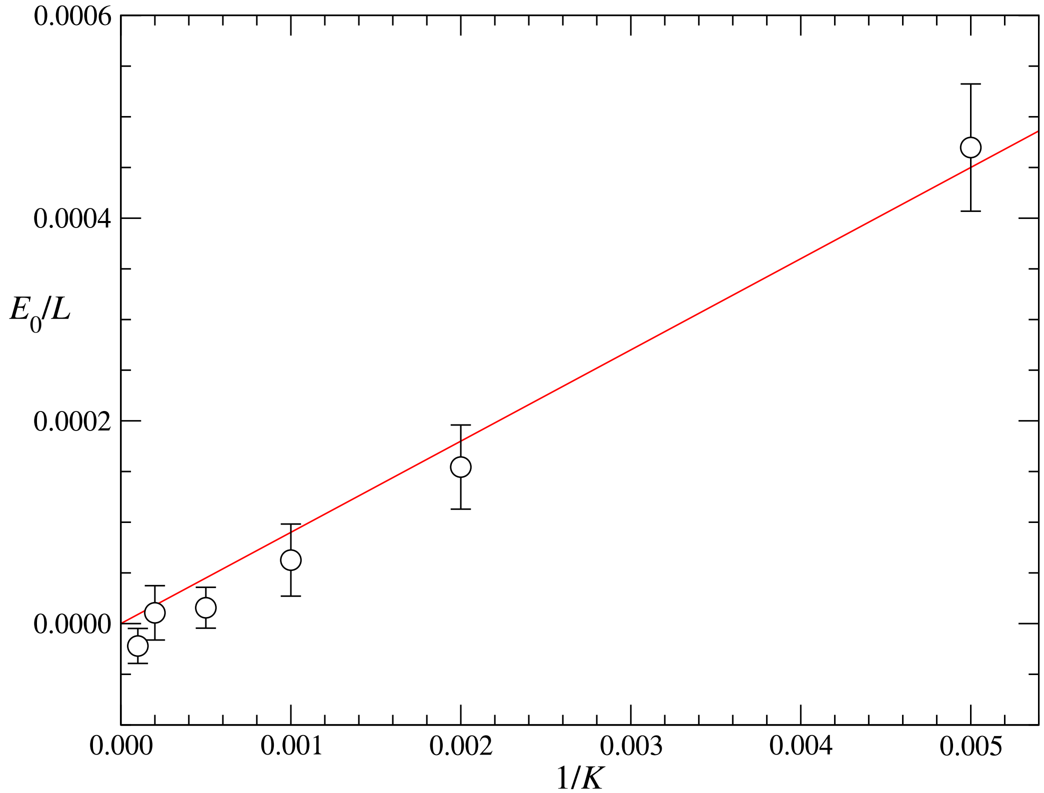
  This conclusion gains further support from the nonvanishing value of the Witten index.

- For even q in strong coupling, the ground state has a positive energy density also for $L \to \infty$ and supersymmetry appears to be broken for all λ_0.

  While weak coupling predicts unbroken supersymmetry when $\lambda_0 < 0$.

Two different approaches used

Numerical results of the first method



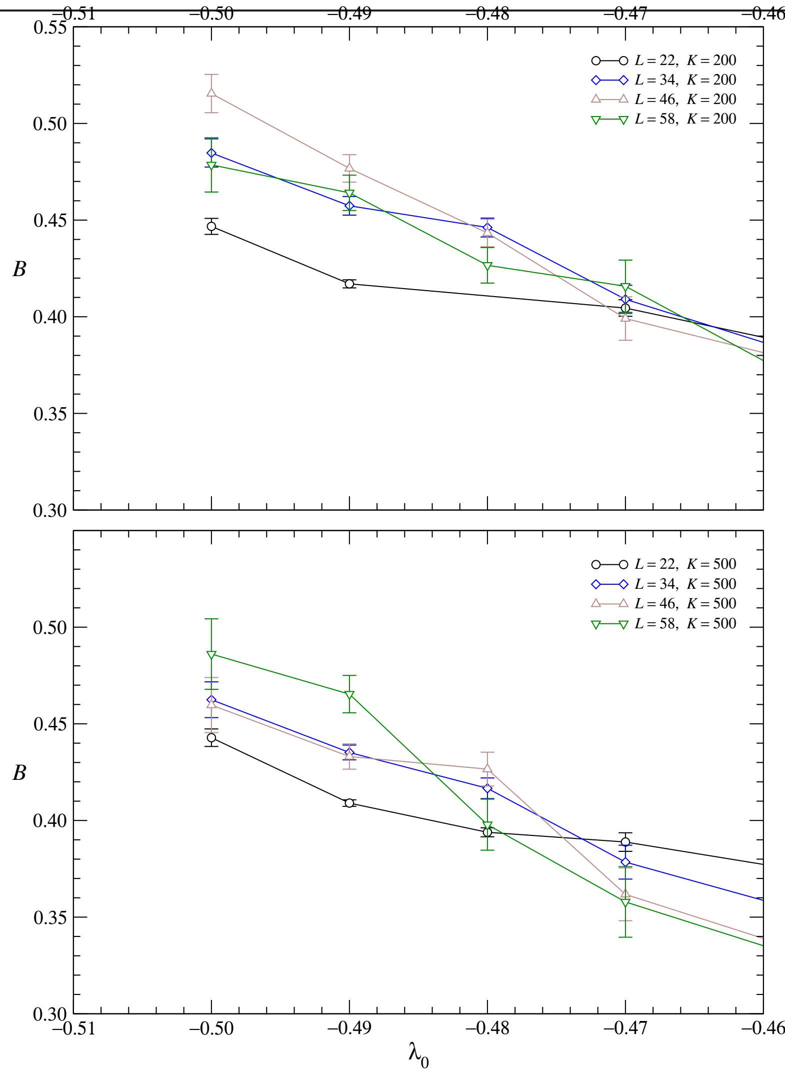
The ground-state energy density E_0/L vs. $1/K$ at $V = \varphi^3$, $L = 22$, with statistics of 1 M iterations for $K < 5000$, 500 k iterations at $K = 5000$, and 300 k iterations at $K = 10000$. (Using the GFMC algorithm).

case $V = \lambda_2 \varphi^2 + \lambda_0$

For fixed $\lambda_2 = 0.5$, we may expect (in the $L \to \infty$ limit) a phase transition at

$\lambda_0 = \lambda_0^{(c)}(\lambda_2)$

separating a phase of broken SUSY (high λ_0) from a phase of unbroken SUSY (low λ_0).



The Binder cumulant B vs. λ_0. $(\lambda_0^{(c)} = -0.48 \pm 0.01)$

The usual technique for the study of a phase transition is the crossing method applied to the Binder cumulant, B. The crossing method consists in plotting B vs. λ_0 for several values of L. The crossing point $\lambda_0^{\mathrm{cr}}(L_1, L_2)$, determined by the condition

$$B(\lambda_0^{\mathrm{cr}}, L_1) = B(\lambda_0^{\mathrm{cr}}, L_2)$$

is an estimator of $\lambda_0^{(c)}$.

The convergence is dominated by the critical exponent ν of the correlation length and by the critical exponent ω of the leading corrections to scaling

$$\lambda_0^{\mathrm{cr}}(L_1, L_2) = \lambda_0^{(c)} + O(L_1^{-\omega-1/\nu}, L_2^{-\omega-1/\nu})$$

we expect the phase transition we are studying to be in the Ising universality class, for which $\nu = 1$ and $\omega = 2$, and therefore we expect fast convergence $\lambda_0^{\mathrm{cr}} \to \lambda_0^{(c)}$.

Numerical results of the second method

The method is based on the calculation of rigorous lower bounds on the ground state energy density in the *infinite-lattice* limit.

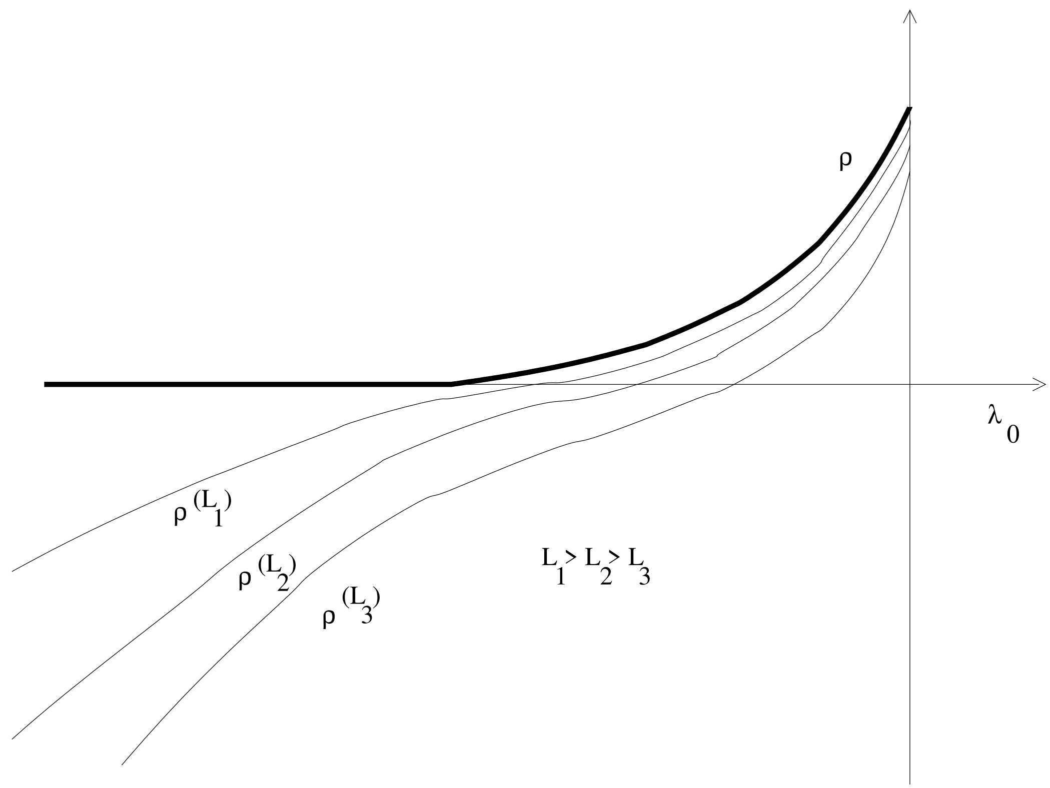
The lattice version of the Wess-Zumino model conserves enough supersymmetry to prove that the ground state has a non negative energy density $\rho \geq 0$, as its continuum limit.

Moreover the ground state is supersymmetric if and only if $\rho = 0$, whereas it breaks (dynamically) supersymmetry if $\rho > 0$. Therefore, if an exact positive lower bound ρ_{LB} is found with $0 < \rho_{\mathrm{LB}} \leq \rho$, we can claim that supersymmetry is broken.

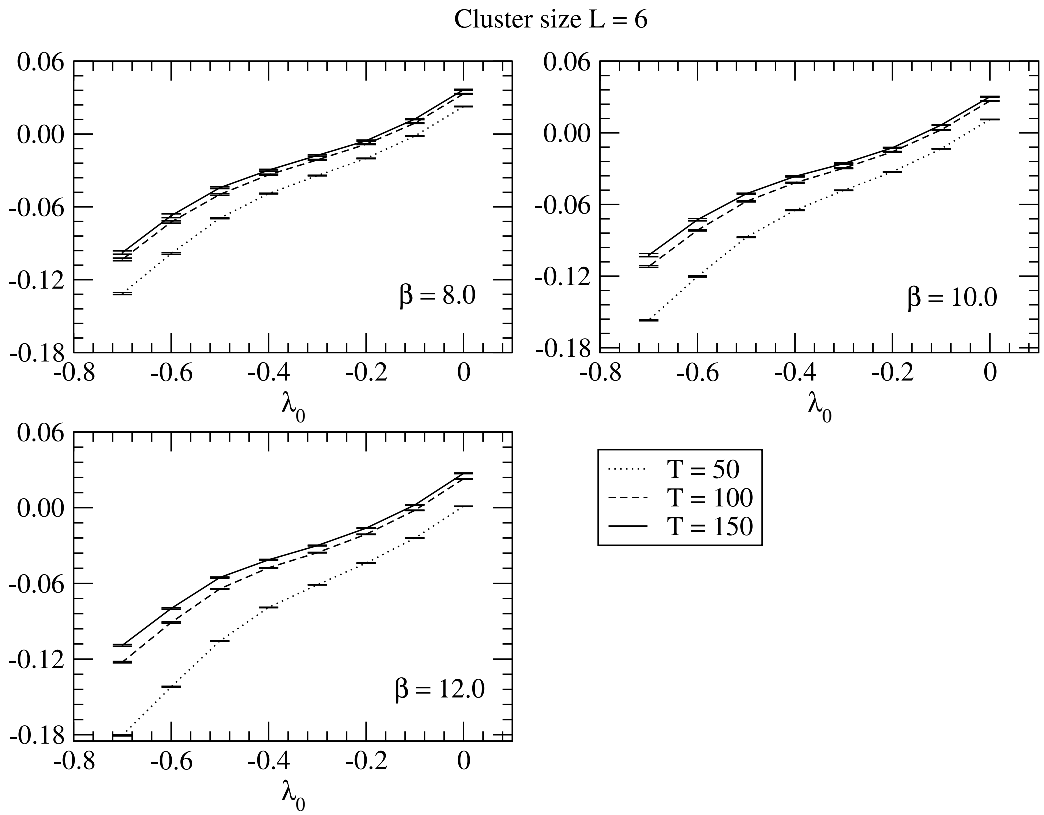
The relevant quantity for our analysis is the ground state energy density ρ evaluated on the infinite lattice

$$\rho = \lim_{L \to \infty} \frac{E_0(L)}{L}$$

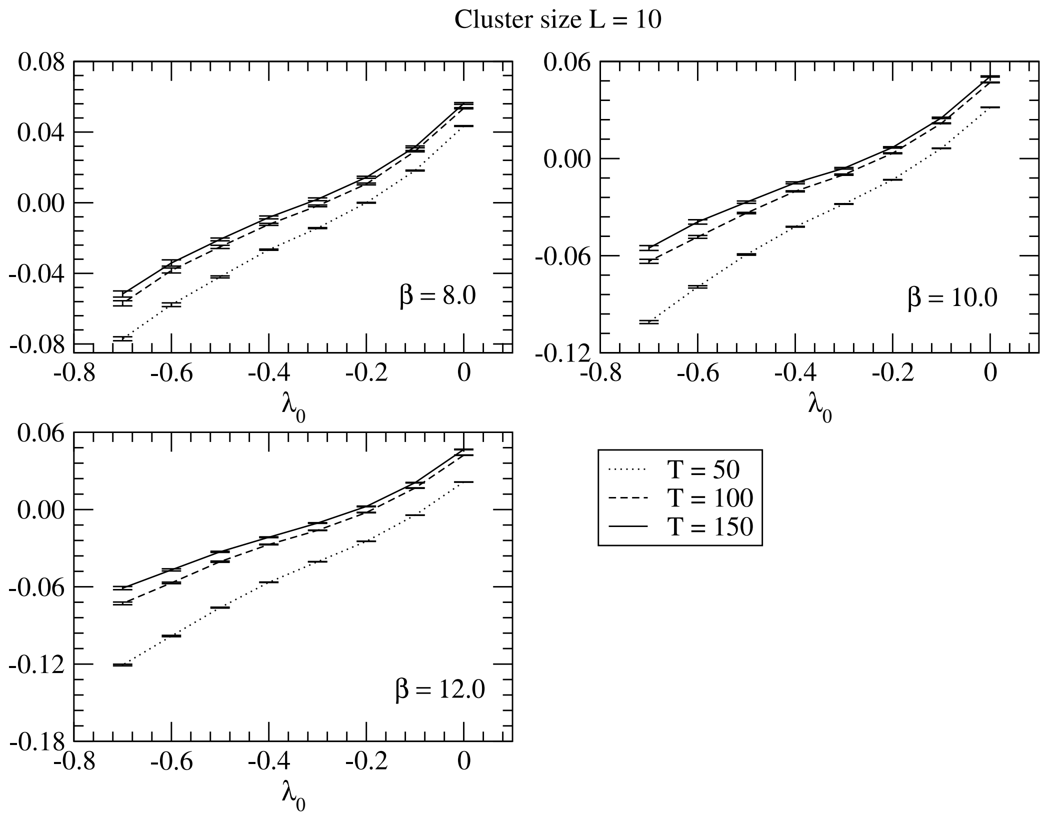
It can be used to tell between the two phases of the model: supersymmetric with $\rho = 0$ or broken with $\rho > 0$.



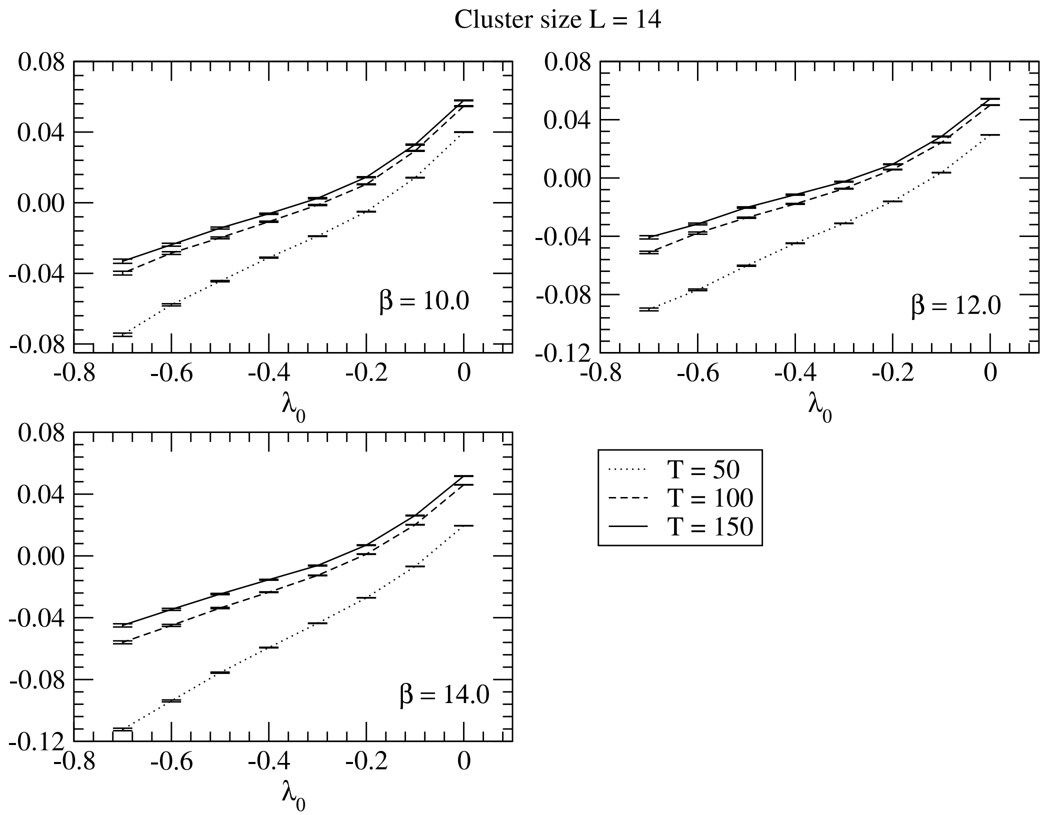
Qualitative plot of the functions $\rho(\lambda_0)$ and $\rho^{(L)}(\lambda_0)$. We see that a single zero is expected in $\rho^{(L)}(\lambda_0)$ at some $\lambda_0 = \lambda_0(L)$. Since $\lim_{L\to\infty} \rho^{(L)} = \rho$, we expect that $\lambda_0(L) \to \lambda_0^*$ for $L \to \infty$ allowing for a determination of the critical coupling λ_0^*.



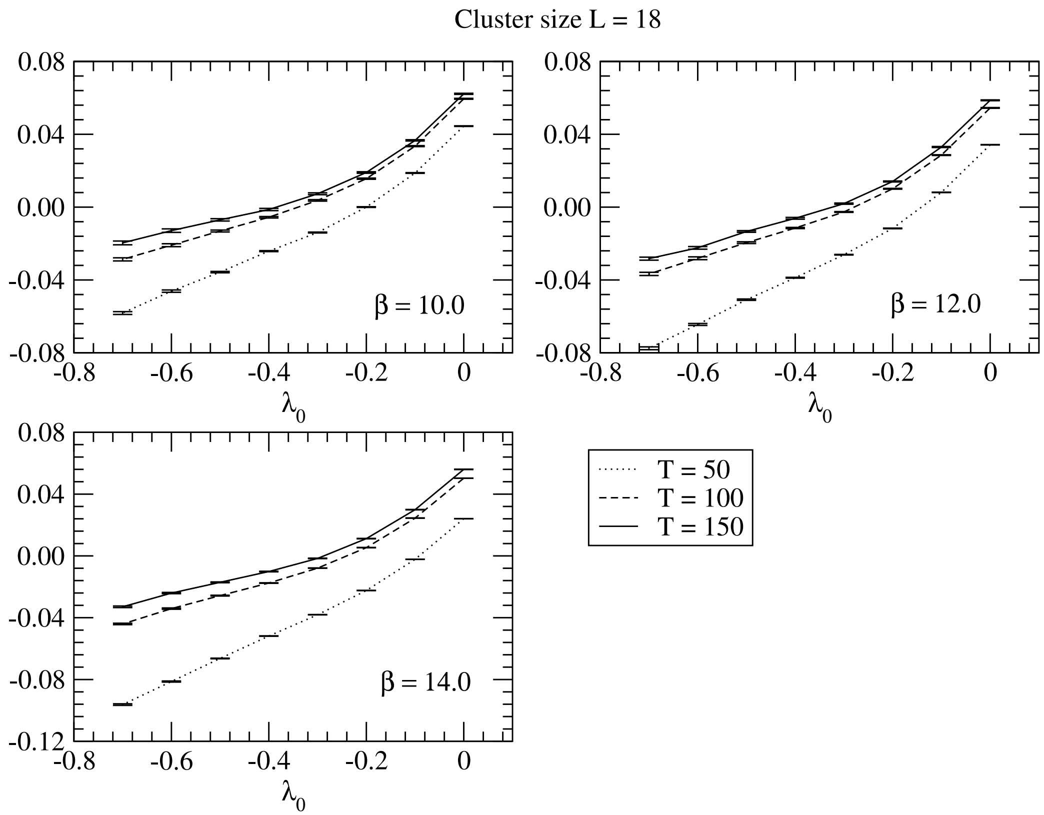
Plot of the energy lower bound $\rho^{(L)}(\beta, T)$ at $L = 6$



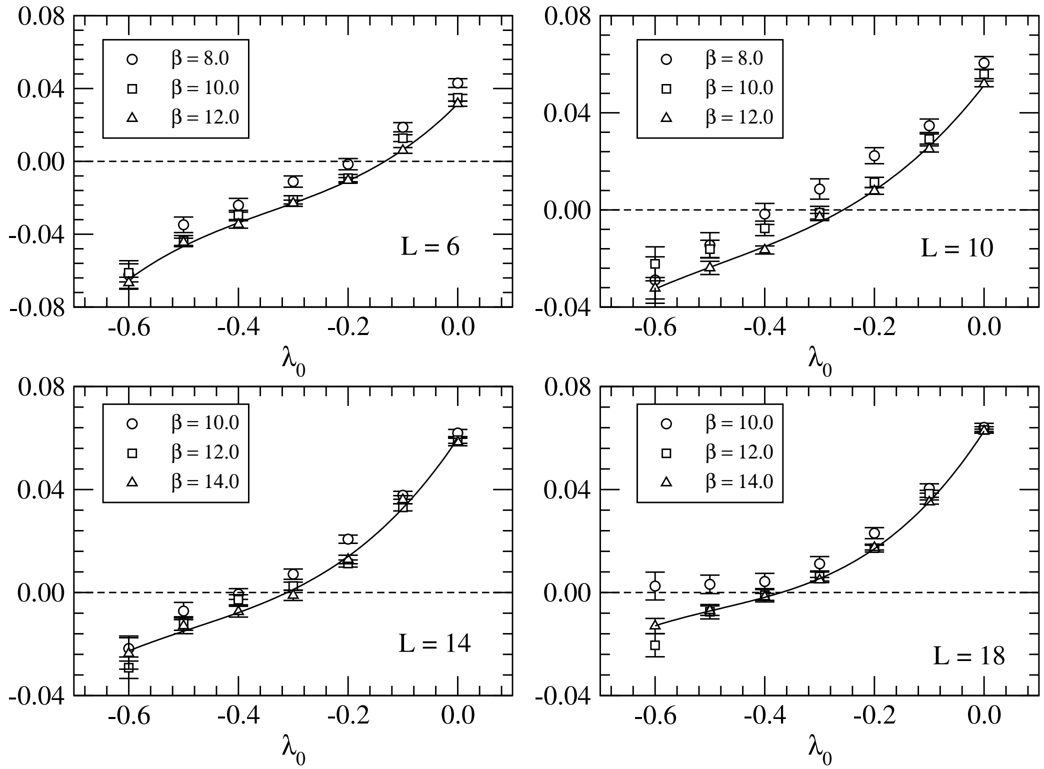
Plot of the energy lower bound $\rho^{(L)}(\beta, T)$ at $L = 10$



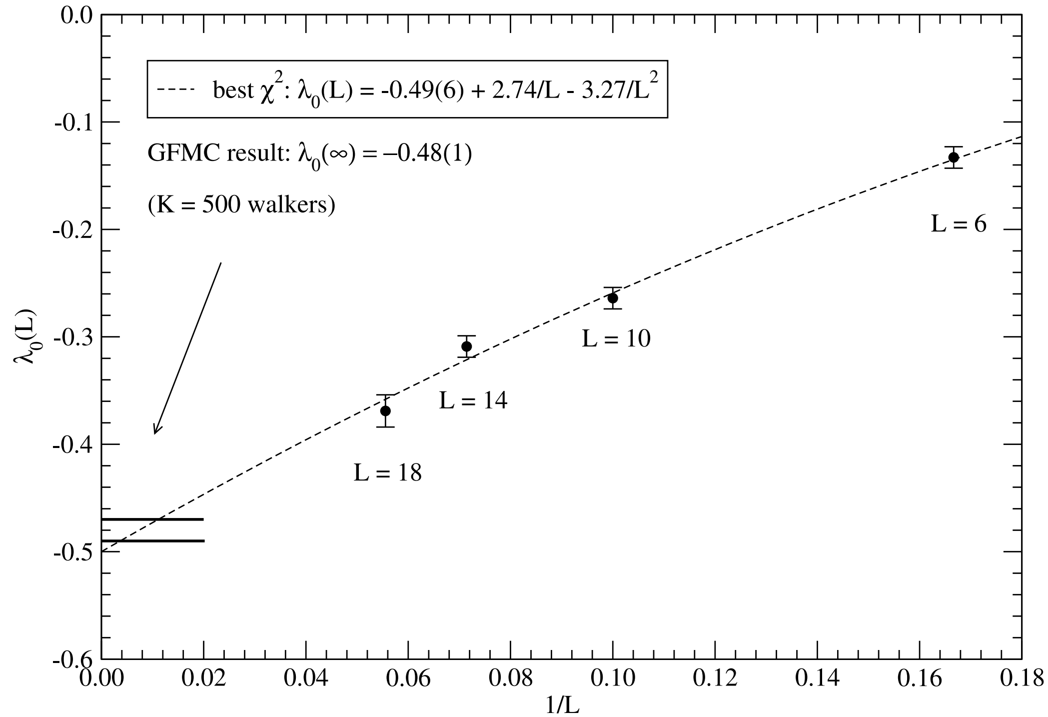
Plot of the energy lower bound $\rho^{(L)}(\beta, T)$ at $L = 14$



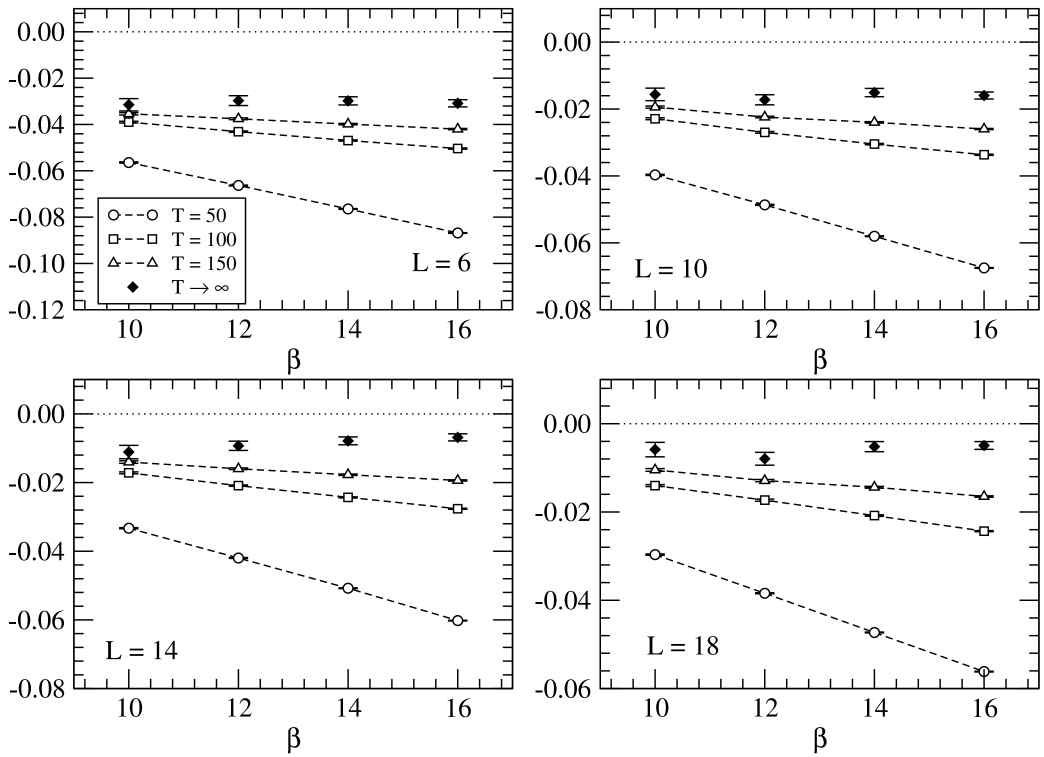
Plot of the energy lower bound $\rho^{(L)}(\beta, T)$ at $L = 18$



Extrapolated bound $\rho^{(L)}(\beta) = \lim_{T\to\infty} \rho^{(L)}(\beta, T)$. As one can see, for each cluster size L, the two curves at the highest values of β almost coincide.



The best fit with a parabolic function gives $\lambda_0^* = -0.49 \pm 0.06$ quite in agreement with the previous $\lambda_{0,\mathrm{GFMC}}^* = -0.48 \pm 0.01$.



Plot of the energy lower bound $\rho^{(L)}(\beta, T)$ at various L, β and T for the cubic prepotential

Conclusions and Outlook

- The Wess-Zumino model is an interesting model to understand how to put GW fermions with exact lattice supersymmetry.
- Study of the Ward-Takahashi identities.
- numerical simulations of this model (at leat in two dimensions)
- The forward step would be to apply to $N = 1$ SYM (more tricky!)
- ...