The QCD Phase Diagram from Lattice Simulations

Simon Hands *University of Wales Swansea*

- The QCD Phase Diagram
- Difficulties at $\mu \neq 0$
- Progress at small μ/T
- Color Superconductivity
- Superfluidity in the NJL Model

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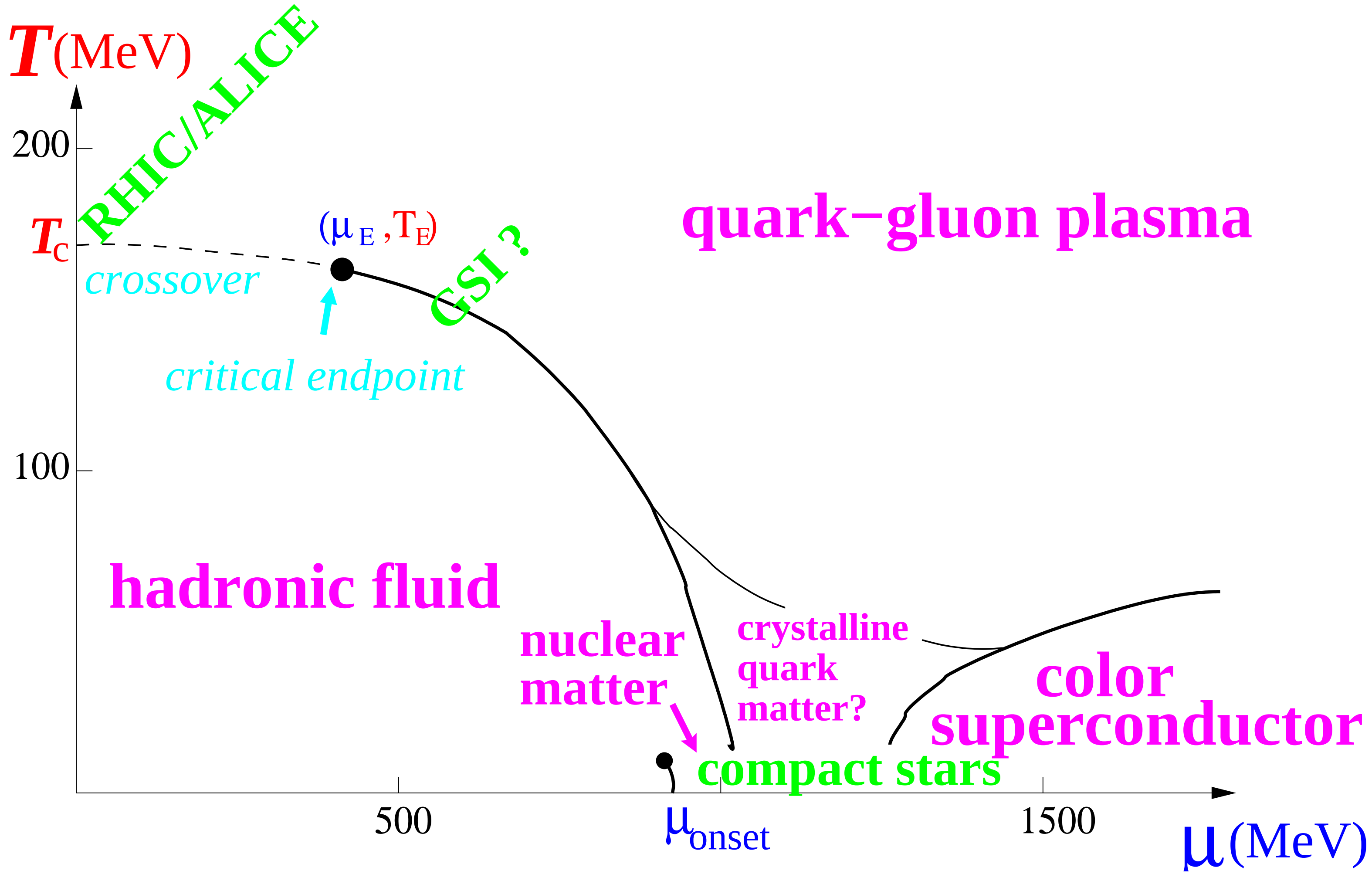
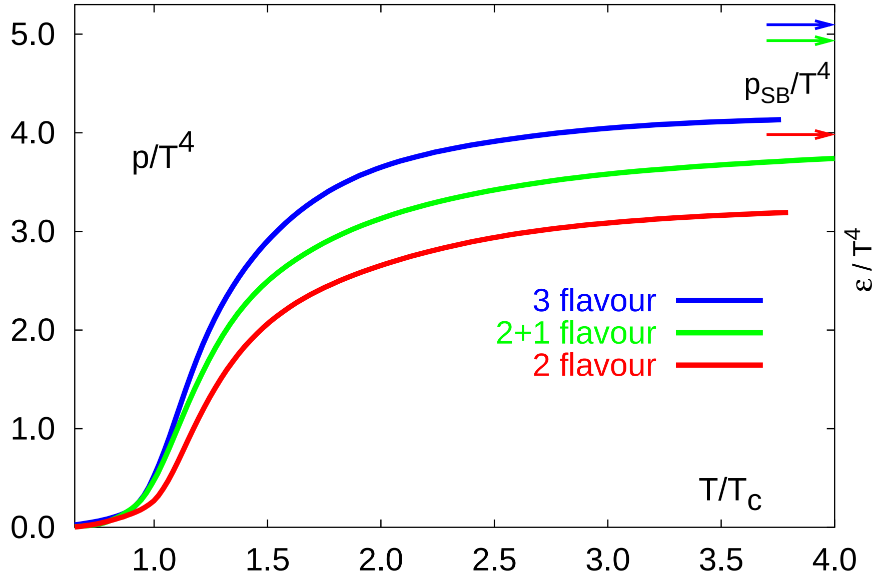
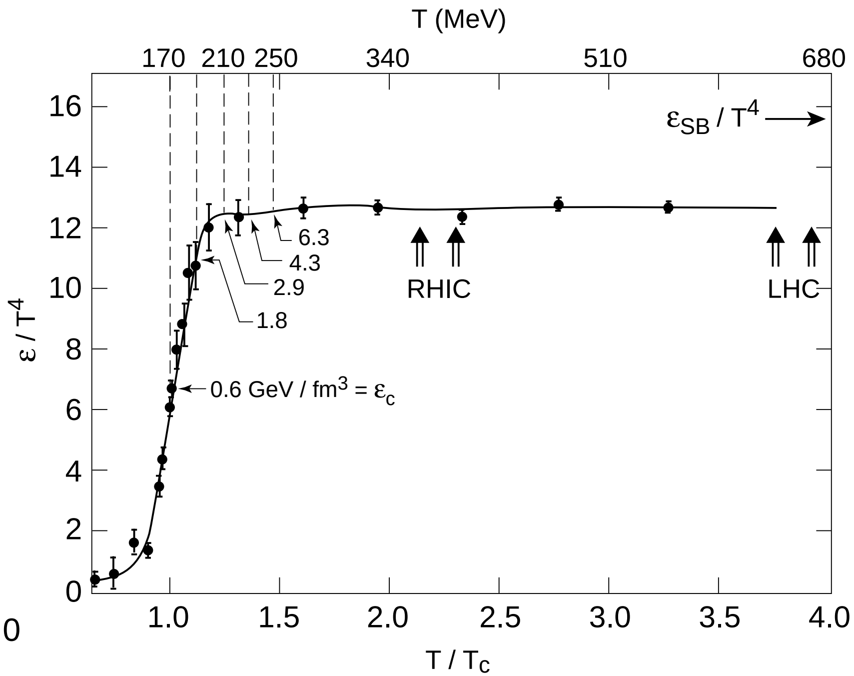
The QCD Phase Diagram



Equation of State at $\mu_B = 0$ ($L_t = 4$)





Bielefeld group (2000)

- For $N_f = 2$ transition is crossover
- For $N_f = 3$ and $m < m_c$ transition is first order
- For realistic "$N_f = 2 + 1$" a crossover is favoured, but more work needed

The Sign Problem for $\mu \neq 0$

In Euclidean metric the QCD Lagrangian reads

$$\mathcal{L}_{QCD} = \bar{\psi}(M + m)\psi + \frac{1}{4}F_{\mu\nu}F_{\mu\nu}$$

with $M(\mu) = D\!\!\!/[A] + \mu\gamma_0$

Straightforward to show $\gamma_5 M(\mu)\gamma_5 \equiv M^\dagger(-\mu) \Rightarrow$

$$\det M(\mu) = (\det M(-\mu))^*$$

ie. Path integral measure is not positive definite for $\mu \neq 0$

Fundamental reason is explicit breaking of time reversal symmetry

Monte Carlo importance sampling, the mainstay of lattice QCD, is ineffective

A formal solution to the Sign Problem is *reweighting* ie. to include the phase of the determinant in the observable:

$$\langle \mathcal{O} \rangle \equiv \frac{\langle\langle \mathcal{O}\, \arg(\det M) \rangle\rangle}{\langle\langle \arg(\det M) \rangle\rangle}$$

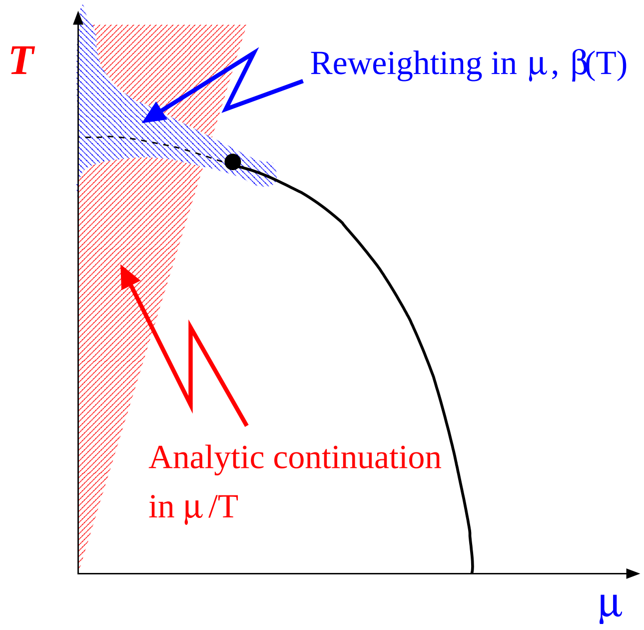
with $\langle\langle \ldots \rangle\rangle$ defined with a positive measure $|\det M| e^{-S_{boson}}$

Unfortunately both denominator and numerator are exponentially suppressed:

$$\langle\langle \arg(\det M) \rangle\rangle = \frac{Z_{true}}{Z_{fake}} = \exp(-\Delta F) \sim \exp(-\# V)$$

Expect signal to be overwhelmed by noise in thermodynamic limit $V \to \infty$

Two Routes into the Plane



(I) Analytic continuation in μ/T by either

Taylor expansion @ $\mu = 0$

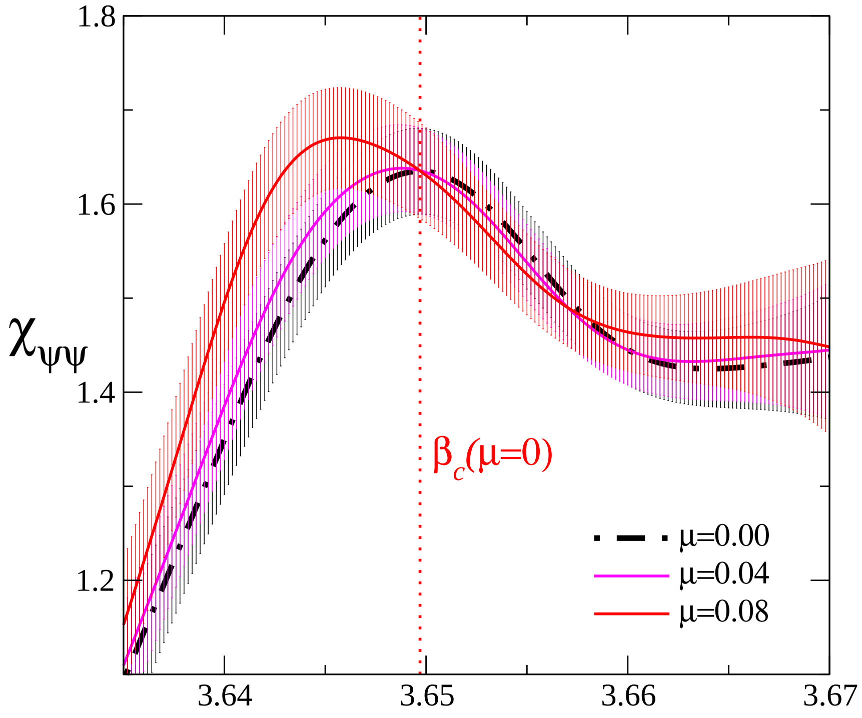
Gavai & Gupta; QCDTARO

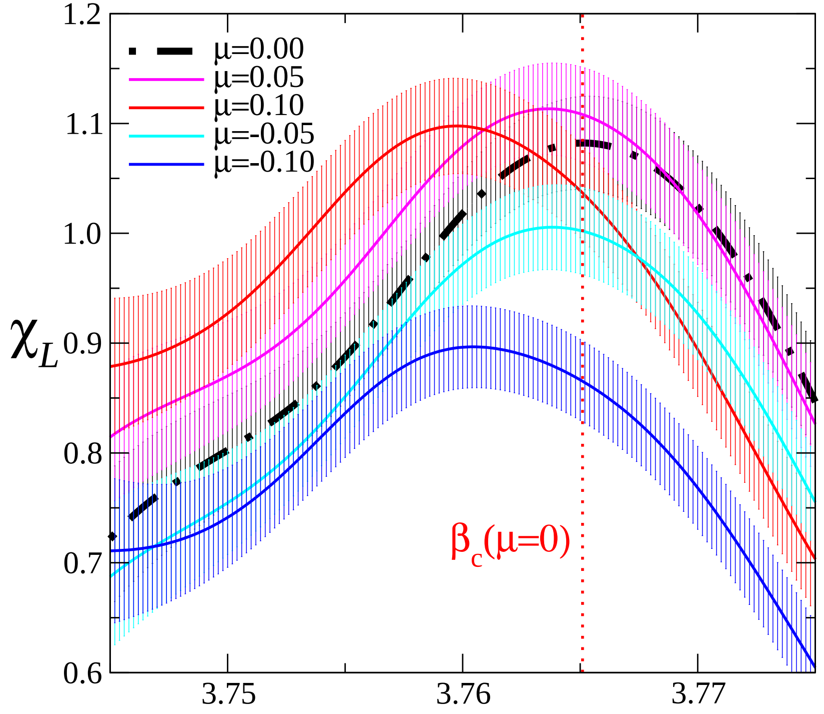
Simulation with imaginary $\tilde{\mu} = i\mu$ de Forcrand & Philipsen; d'Elia & Lombardo

Effective for $\frac{\mu}{T} <$ min $\left(\frac{\mu_E}{T_E}, \frac{\pi}{3}\right)$

(II) Reweighting along transition line $T_c(\mu)$ Fodor & Katz

Overlap between (μ, T) and $(\mu + \Delta\mu, T + \Delta T)$ remains large, so multi-parameter reweighting unusually effective





The Bielefeld/Swansea group uses a hybrid approach; ie. we reweight using a Taylor expansion of the weight:

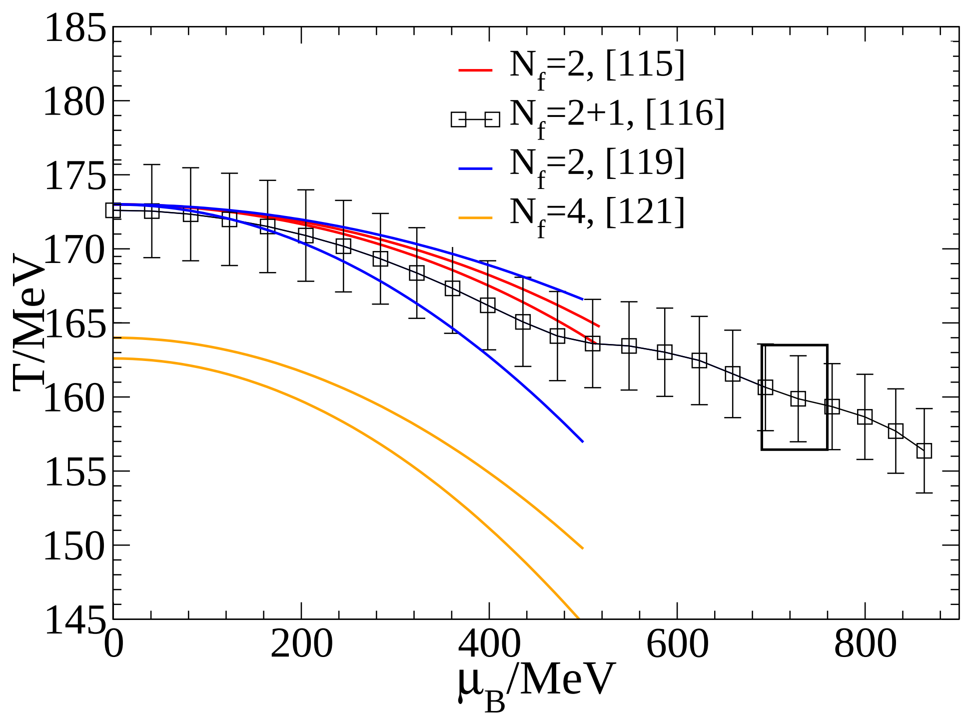
Allton *et al*, PRD66(2002)074507

$$\ln\left(\frac{\det M(\mu)}{\det M(0)}\right) = \sum_n \frac{\mu^n}{n!} \frac{\partial^n \ln \det M}{\partial \mu^n}\bigg|_{\mu=0}$$

This is relatively cheap and enables the use of large spatial volumes ($16^3 \times 4$ using $N_f = 2$ flavors of p4-improved staggered fermion).

Note with $L_t = 4$ the lattice is coarse: $a^{-1}(T_c) \simeq 700$MeV

The (Pseudo)-Critical Line

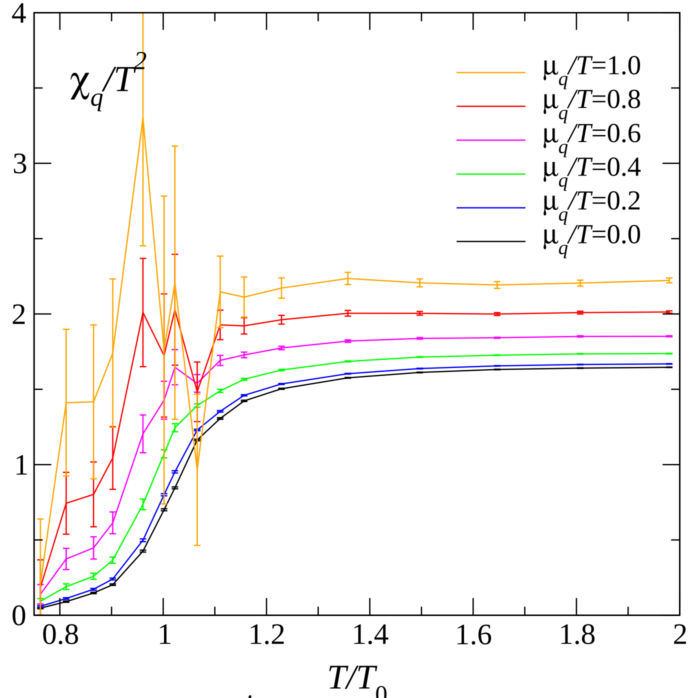


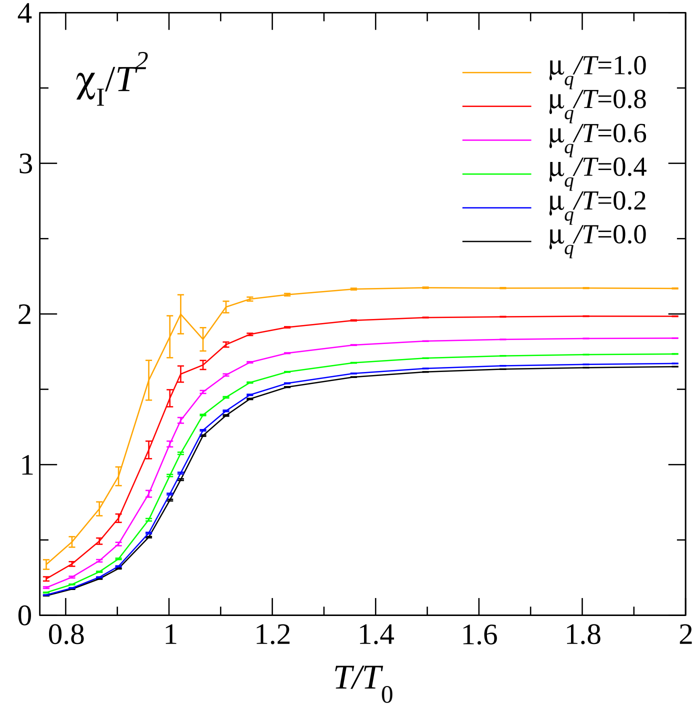
[E. Laermann & O. Philipsen, Ann.Rev.Nucl.Part.Sci.53:163,2003]

Remarkable consensus on the curvature…

RHIC collisions operate in region $\mu_B \sim 45$MeV

Growth of Baryonic Fluctuations





Near $\mu_q/T \sim 1$

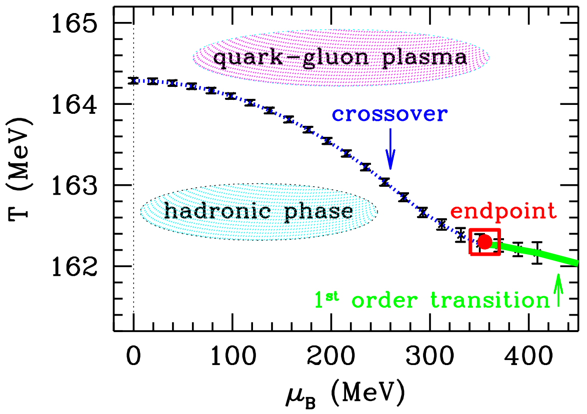
Allton *et al* PRD68(2003)014507

Quark number susceptibility $\chi_q = \frac{\partial^2 \ln Z}{\partial \mu_q^2}$ is singular,

in contrast to isospin susceptibility $\chi_I = \frac{\partial^2 \ln Z}{\partial \mu_I^2}$

Massless field at critical point a combination of the Galilean scalars $\bar{\psi}\psi$ and $\bar{\psi}\gamma_0\psi$?

The Critical Endpoint μ_E/T_E

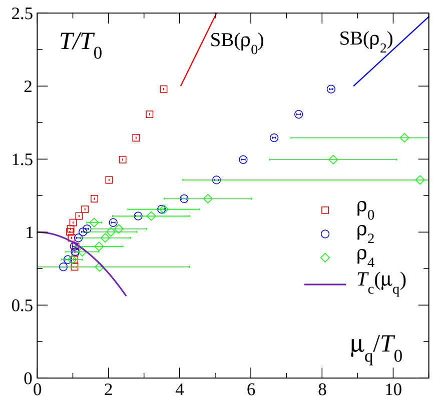


Reweighting estimate
via Lee-Yang zeroes
$\mu_E/T_E = 2.2(2)$
Z. Fodor & S.D. Katz JHEP0404(2004)050

Taylor expansion estimate from apparent radius of convergence

$\mu_E/T_E \gtrsim |c_4/c_6| \sim 3.3(6)$

Allton *et al* PRD68(2003)014507



Analytic estimate via Binder cumulant $\langle(\delta\mathcal{O})^4\rangle/\langle(\delta\mathcal{O})^2\rangle^2$
evaluated at imaginary $\mu \Rightarrow \mu_E/T_E \sim O(20)$!

P. de Forcrand & O. Philipsen NPB673(2003)170

Partial Summary

Approaches with different systematics are yielding encouraging agreement on the critical line $T_c(\mu)$

for small μ/T

Still no consensus on location of the critical endpoint

Need better control over statistics, and over sensitivity to strange quark mass m_s

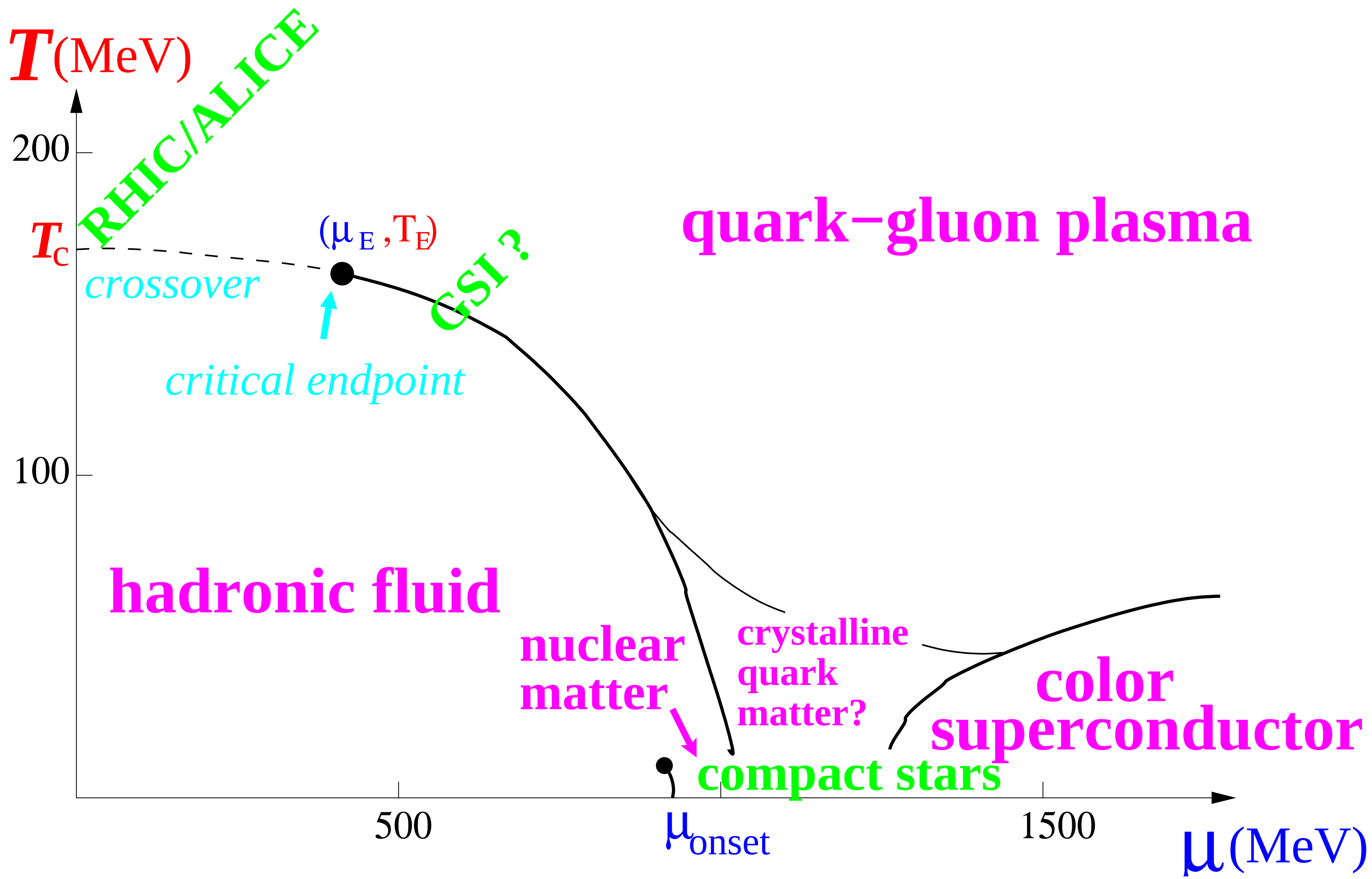
even at $\mu = 0$, estimates of critical quark mass with $N_f = 3$ show strong cutoff-dependence: eg.

$$m_\pi^{crit} = \begin{cases} 290(20)\mathrm{MeV} & \text{standard action} \\ 70(20)\mathrm{MeV} & \text{p4 improved} \end{cases}$$

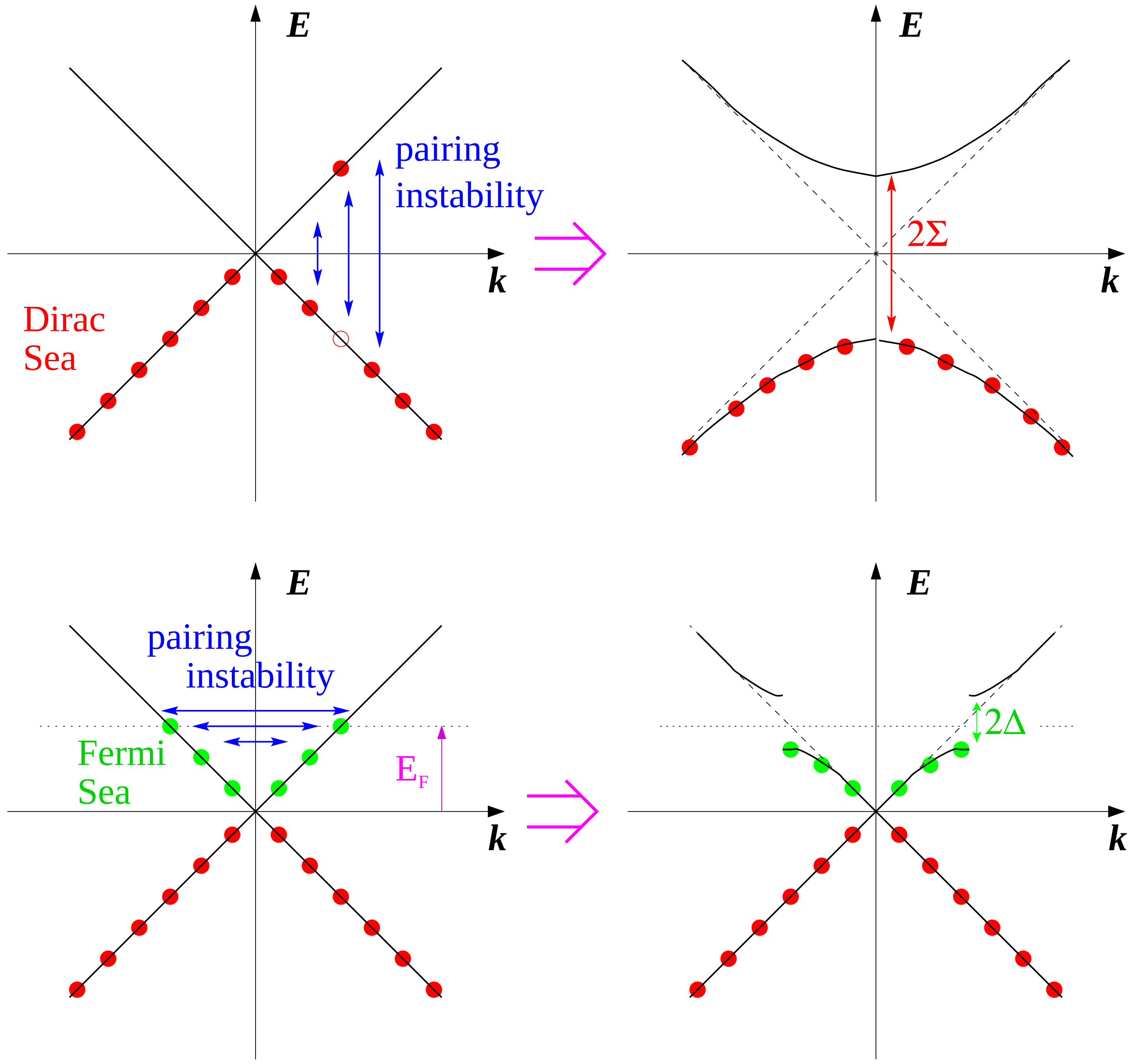
We need to get closer to the continuum limit!

NO OBVIOUS OBSTACLE to calculation of μ_E/T_E

The QCD Phase Diagram



χSB vs. Cooper Pairing



Color Superconductivity

In the asymptotic limit $\mu \to \infty$, $g(\mu) \to 0$, the ground state of QCD is the *color-flavor locked* (CFL) state characterised by a BCS instability, [D. Bailin and A. Love, Phys.Rep. 107(1984)325]
ie. diquark pairs at the Fermi surface condense via

$$\langle q_i^\alpha(p) C\gamma_5 q_j^\beta(-p)\rangle \sim \varepsilon^{A\alpha\beta}\varepsilon_{Aij} \times \text{const.}$$

breaking SU(3)$_c\otimes$SU(3)$_L\otimes$SU(3)$_R$ $\otimes$U(1)$_B$ $\otimes$U(1)$_Q$
$\longrightarrow$SU(3)$_\Delta\otimes$U(1)$_{\tilde{Q}}$

The ground state is simultaneously
superconducting (8 gapped gluons, ie. get mass $O(\Delta)$),
superfluid (1 Goldstone),
and *transparent* (all quasiparticles with $\tilde{Q} \neq 0$ gapped).

[M.G. Alford, K. Rajagopal and F. Wilczek, Nucl.Phys.B537(1999)443]

At smaller densities such that $\mu/3 \sim k_F \lesssim m_s$, expect pairing between u and d only $\Rightarrow$ "2SC" phase

$$\langle q_i^\alpha(p) C\gamma_5 q_j^\beta(-p)\rangle \sim \varepsilon^{\alpha\beta 3}\varepsilon_{ij} \times \text{const.}$$

SU(3)$_c$ $\longrightarrow$ SU(2)$_c$ $\Rightarrow$ 5/8 gluons get gapped
Global SU(2)$_L$$\otimes$SU(2)$_R$$\otimes$U(1)$_B$ unbroken

Another possibility in isopsin asymmetric matter is the so-called "LOFF" phase:

$$\langle u(k_F^u;\uparrow) d(-k_F^d;\downarrow)\rangle \neq 0$$

In the electrically-neutral matter expected in compact stars, $k_F^d - k_F^u = \mu_e \sim 100$MeV $\Rightarrow$ $\langle\psi\psi\rangle$ condensate has $\vec{k} \neq 0$ breaking translational invariance $\Rightarrow$ *crystallisation*

Other ideas:

a 2SC/normal mixed phase (plates? rods?)

or a gapless 2SC where $\langle qq\rangle \neq 0$ but $\Delta = 0$

What can we say at smaller densites $\mu \sim O$(1 GeV) where weak coupling methods can't be trusted? Lattice QCD simulations can't help due to the Sign Problem

In many body theory there are two tractable limits:

	Strong Coupling	Weak Coupling
physical d.o.f.'s	tightly-bound bosons	weakly interacting fermions
superfluidity mechanism	Bose Einstein Condensation	BCS condensation
QFT example	Two Color QCD	NJL model

Both model QFT's can be studied with $\mu \neq 0$ using lattice simulations which evade the Sign Problem.

High-T_c superconducting compounds, cold atoms near a Feshbach resonance, and perhaps QCD, are difficult problems because they belong to neither limit

Gross-Neveu model…

$$\mathcal{L} = \sum_{i=1}^{N_f} \bar{\psi}_i(\partial\!\!\!/ + m + \mu\gamma_0)\psi_i - \frac{g^2}{2N_f}(\bar{\psi}_i\psi_i)^2,$$

…just about the simplest QFT with fermions.
The fundamental interaction is *attractive*
Can also write in terms of an auxiliary scalar σ:

$$\mathcal{L} = \bar{\psi}_i(\partial\!\!\!/ + m + \mu\gamma_0 + \frac{g}{\sqrt{N_f}}\sigma)\psi_i + \frac{1}{2}\sigma^2.$$

For $g^2 > g_c^2 \sim O(\Lambda^{-1})$ and $\mu = 0$ the ground state has a dynamically-generated fermion mass $\Sigma_0 = \frac{g}{\sqrt{N_f}}\langle\sigma\rangle \neq 0$ given in the $N_f \to \infty$ limit by the chiral *Gap Equation*

$$\Sigma_0 = g^2 \text{tr} \int_p \frac{1}{ip\!\!\!/ + \Sigma_0}$$

In same limit σ acquires non-trivial dynamics:

$$D_\sigma^{-1}(k^2) = 1 - \Pi(k^2) \propto \begin{cases} k^2 + 4\Sigma_0^2 & k \ll \Sigma_0 \\ k^{d-2} & k \gg \Sigma_0 \end{cases}$$

$\Rightarrow$ For $2 < d < 4$ model is unexpectedly *renormalisable*

ie. GN model has an UV-stable renormalisation group fixed point and an interacting continuum limit as $g \to g_c$.

Wilson (1974)

In 2+1d GN can be regarded as a fundamental QFT

but without gluons or confinement

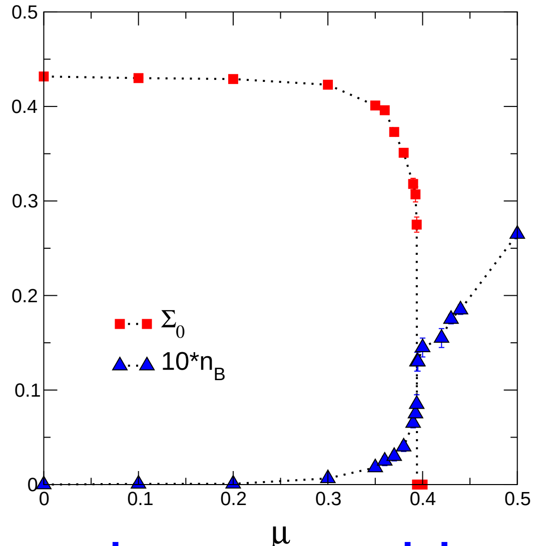
In 3+1d this property ceases to hold, and the GN model (like NJL) must be regarded as an effective field theory requiring an explicit UV cutoff.

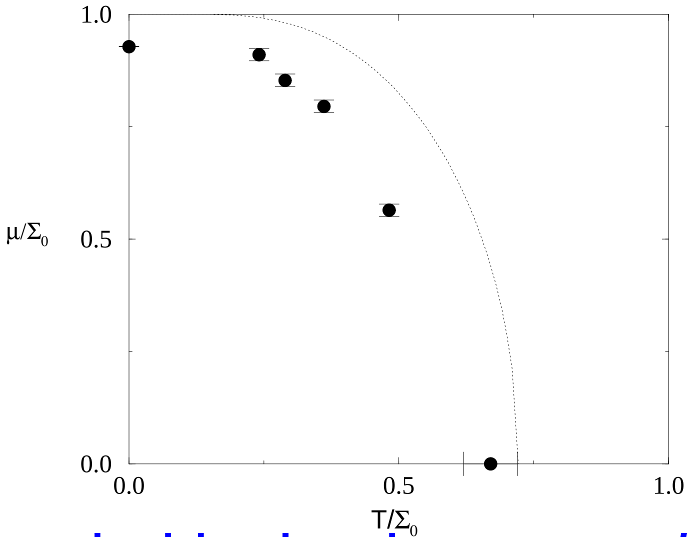
GN Thermodynamics

The large-N_f approach can also to be applied to $T, \mu \neq 0$ and predicts a chiral symmetry restoring phase transition:

$$T_c|_{\mu=0} = \frac{\Sigma_0}{2\ln 2}; \quad \mu_c|_{T=0} = \Sigma_0$$

Lattice simulations can model $N_f < \infty$ even for $\mu \neq 0$

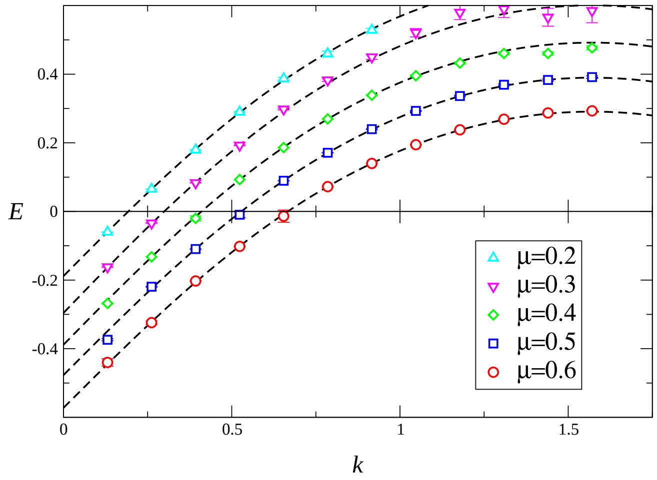




There is even evidence for a tricritical point at *small* $\frac{T}{\mu}$*!*

[J.B. Kogut and C.G. Strouthos PRD63(2001)054502]

Fermion Dispersion relation



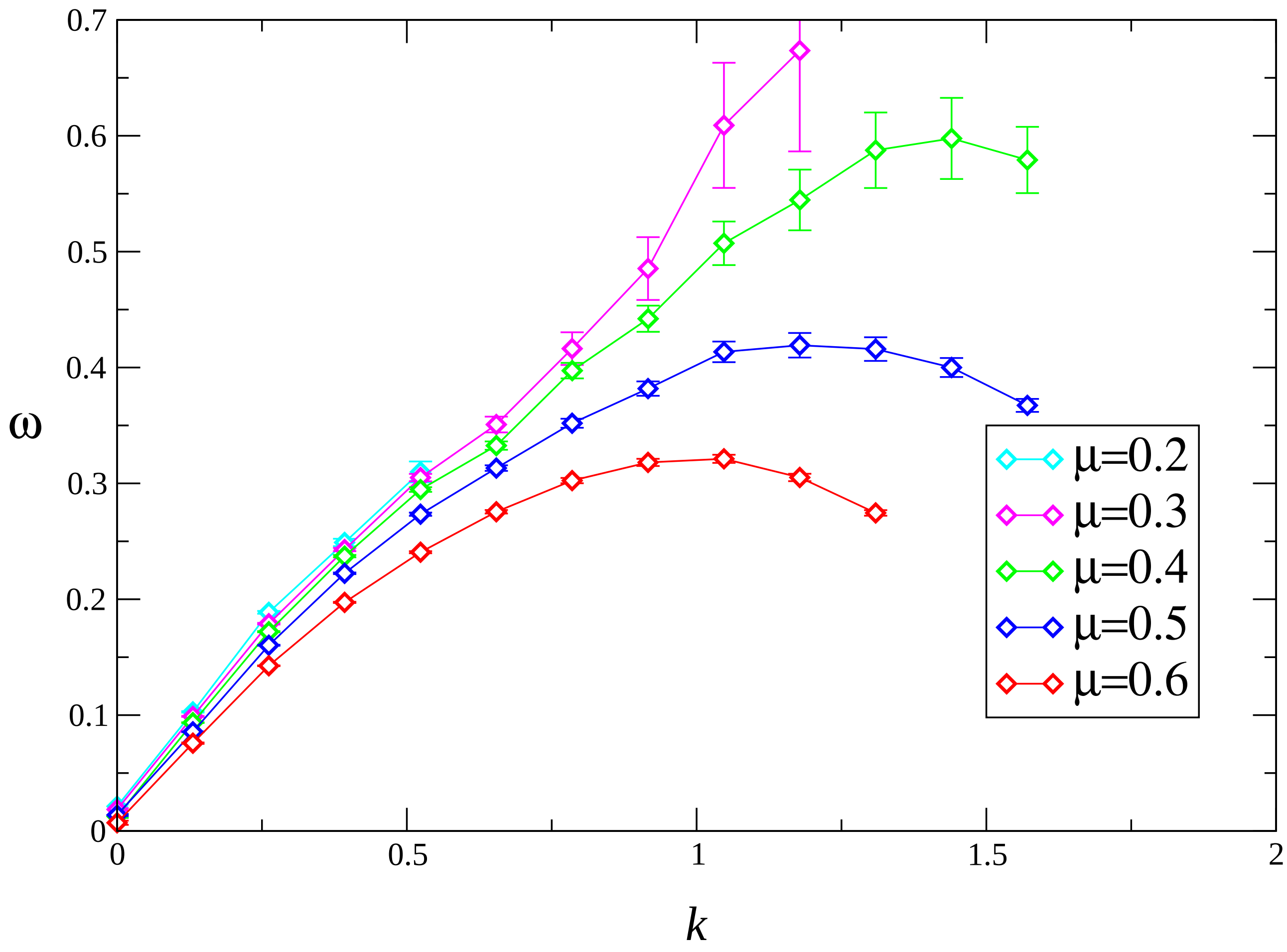
μ	K_F	β_F	$K_F/\mu\beta_F$
0.2	0.190(1)	0.989(1)	0.962(5)
0.3	0.291(1)	1.018(1)	0.952(4)
0.4	0.389(1)	0.999(1)	0.973(1)
0.5	0.485(1)	0.980(1)	0.990(2)
0.6	0.584(3)	0.973(1)	1.001(2)

The fermion dispersion relation is fitted with

$$E(|\vec{k}|) = -E_0 + D\sinh^{-1}(\sin|\vec{k}|)$$

yielding the Fermi liquid parameters

$$K_F = \frac{E_0}{D}; \qquad \beta_F = D\frac{\cosh E_0}{\cosh K_F}$$



Dispersion relation $\omega(|\vec{k}|)$ extracted from meson channel interpolated by an operator $\bar{\psi}(\gamma_0 \otimes \tau_2)\psi$

A massless vector excitation?

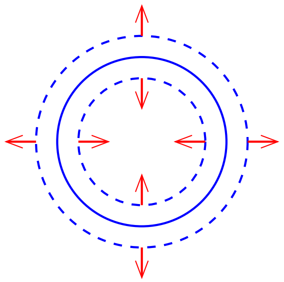
SJH & C.G. Strouthos PRD70(2004)056006

Sounds Unfamiliar?

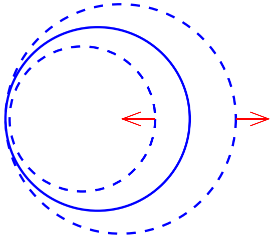
Light vector states in medium are of of great interest:

Brown-Rho scaling, vector condensation. . .

In the Fermi liquid framework a possible explanation is a *collective excitation* thought to become important as $T \to 0$: *Zero Sound*



Ordinary FIRST sound is a breathing mode of the Fermi surface: velocity $\beta_1 \simeq \frac{1}{\sqrt{2}} \frac{k_F}{\mu}$



ZERO sound is a propagating distortion of the Fermi surface: velocity $\beta_0 \sim \beta_F$ must be determined self-consistently

The NJL Model

Effective description of soft pions interacting with nucleons

$$\mathcal{L}_{NJL} = \bar\psi(\partial\!\!\!/ + m + \mu\gamma_0)\psi - \frac{g^2}{2}[(\bar\psi\psi)^2 - (\bar\psi\gamma_5\vec\tau\psi)^2]$$
$$\sim \bar\psi(\partial\!\!\!/ + m + \mu\gamma_0 + \sigma + i\gamma_5\vec\pi.\vec\tau)\psi + \frac{2}{g^2}(\sigma^2 + \vec\pi.\vec\pi)$$

Introduce isopsin indices so full global symmetry is SU(2)$_L\otimes$SU(2)$_R\otimes$U(1)$_B$

Dynamical χSB for $g^2 > g_c^2 \Rightarrow$ isotriplet Goldstone $\vec\pi$

Scalar isoscalar diquark $\psi^{tr}C\gamma_5\otimes\tau_2\otimes A^{color}\psi$ breaks U(1)$_B$

$\Rightarrow$ diquark condensation signals high density ground state is superfluid

Model is renormalisable in 2+1d so GN analysis holds

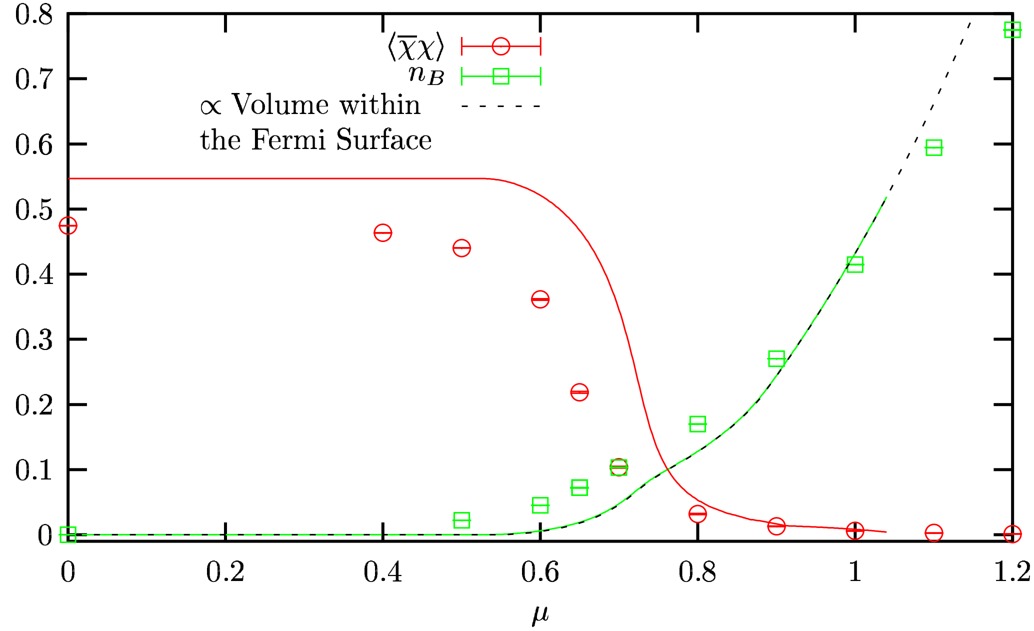
In 3+1d, an explicit cutoff is required. We follow the large-N_f (Hartree) approach of Klevansky (1992) and match lattice parameters to low energy phenomenology:

Phenomenological Observables fitted	Lattice Parameters extracted
$\Sigma_0 = 400$MeV	$ma = 0.006$
$f_\pi = 93$MeV	$1/g^2 = 0.495$
$m_\pi = 138$MeV	$a^{-1} = 720$MeV

The lattice regularisation preserves

SU(2)$_L\otimes$SU(2)$_R\otimes$U(1)$_B$

Equation of State and Diquark Condensation



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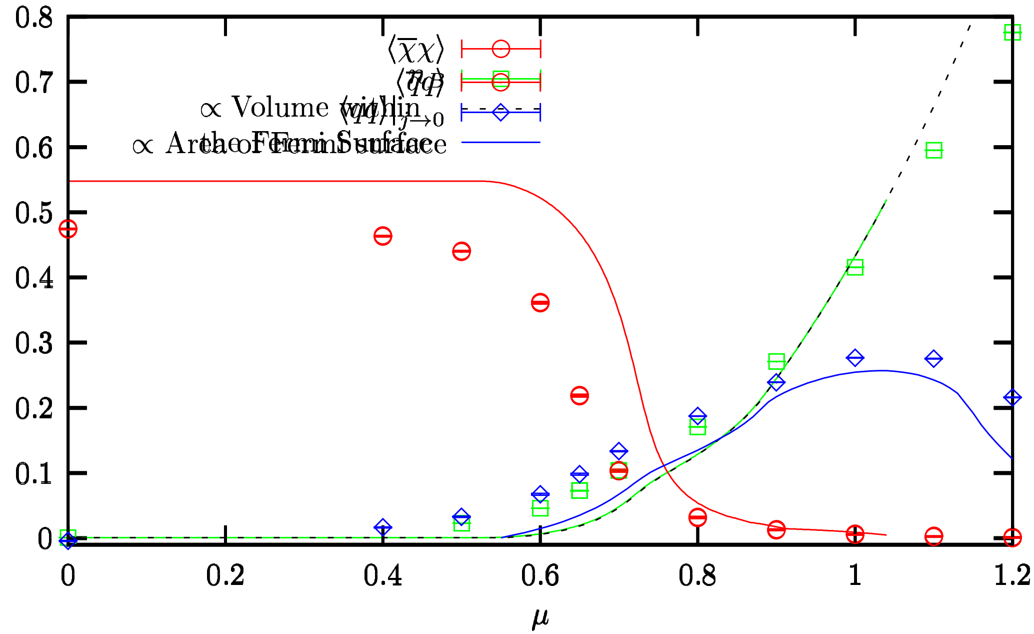
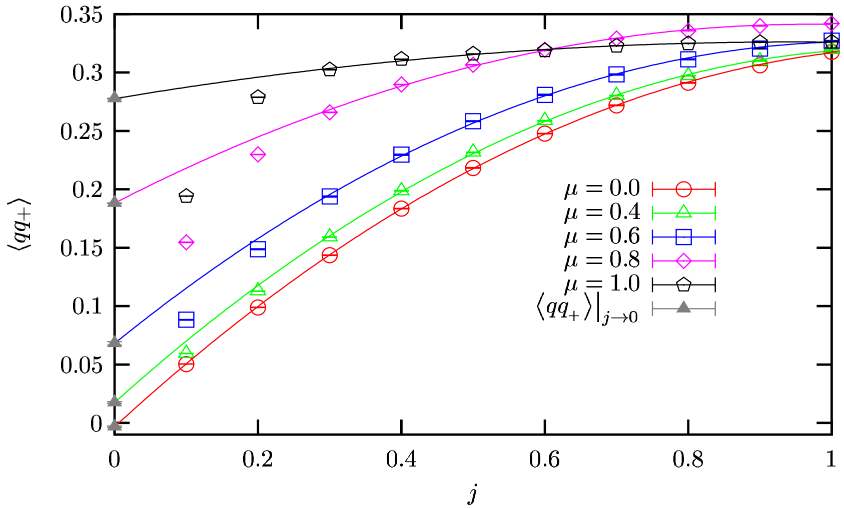




Add source $j[\psi^{tr}\psi + \bar{\psi}\bar{\psi}^{tr}]$

Diquark condensate estimated by taking $j \to 0$

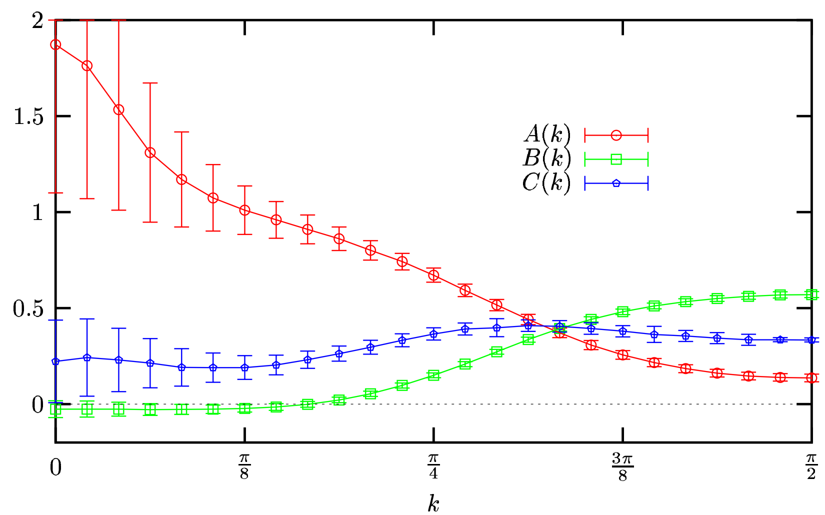
Our fits exclude $j \leq 0.2$

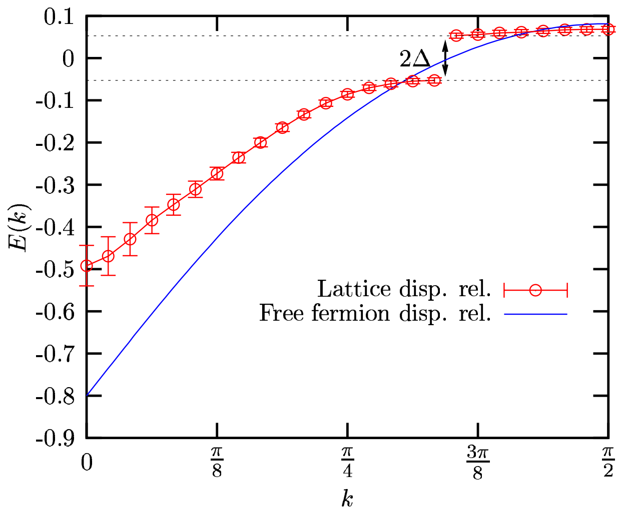
The Superfluid Gap

Quasiparticle propagator:

$$\langle \psi_u(0)\bar{\psi}_u(t)\rangle = Ae^{-Et} + Be^{-E(L_t-t)}$$
$$\langle \psi_u(0)\psi_d(t)\rangle = C(e^{-Et} - e^{-E(L_t-t)})$$

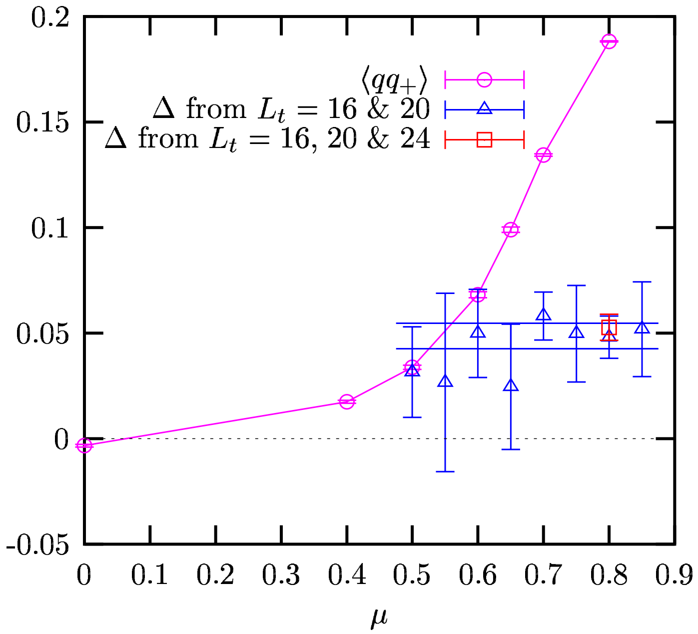
Results from $96 \times 12^2 \times L_t$, $\mu a = 0.8$ extrapolated to $L_t \to \infty$ (ie. $T \to 0$) then $j \to 0$





The gap at the Fermi surface signals superfluidity

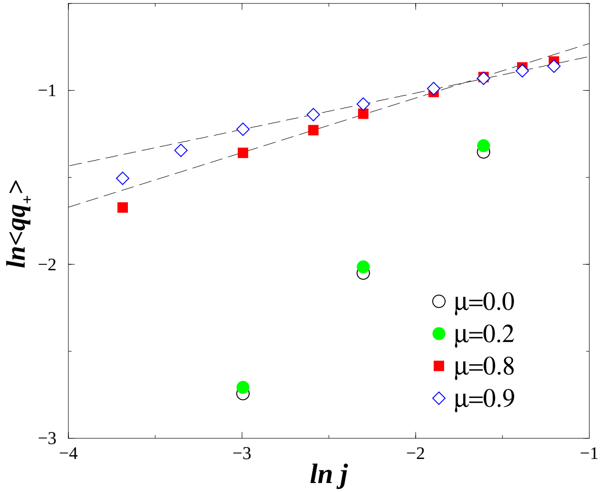
SJH & D.N. Walters PLB548(2002)196 PRD69(2004)076011

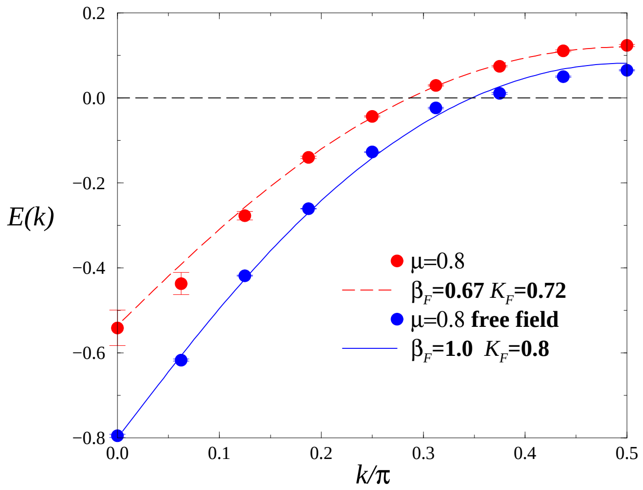


- Near transition, $\Delta \sim$ const, $\langle \psi\psi \rangle \sim \Delta\mu^2$
- $\Delta/\Sigma_0 \simeq 0.15 \Rightarrow \Delta \simeq 60$MeV
  in agreement with self-consistent approaches
- $\Delta/T_c = 1.764$ (BCS) $\Rightarrow L_{tc} \sim 35$
  explains why $j \to 0$ limit is problematic
- Currently studying $\mu_I = (\mu_u - \mu_d) \neq 0$,
  which “re”introduces a sign problem!

NJL Model in 2+1d

SJH, B. Lucini & S.E. Morrison PRL86(2001)753 PRD65(2002)036004





Condensate $\langle\psi\psi\rangle \propto j^{\frac{1}{\delta}}$ No gap at Fermi surface

High density phase $\mu > \mu_c$ is *critical*, rather like the low-T phase of the 2d XY model Kosterlitz & Thouless (1973)

$\delta = \delta(\mu) \simeq 3 - 5$ Cf. 2d XY model $\delta \geq 15$

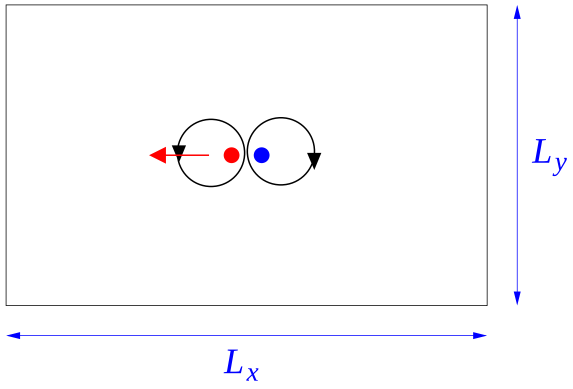
New universality class due to massless fermions

No long-range ordering, but phase coherence

$\langle\psi\psi(0)\psi\psi(r)\rangle \propto r^{-\eta(\mu)} \Rightarrow$ *Thin Film Superfluidity*

Persistent Flow in a $2d$ Superfluid

If $\theta(x)$ is the local phase of the condensate, then the supercurrent $\vec{J}_s = \Upsilon \vec{\nabla}\theta$



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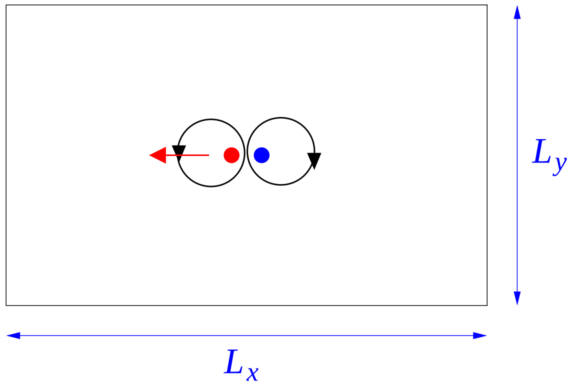

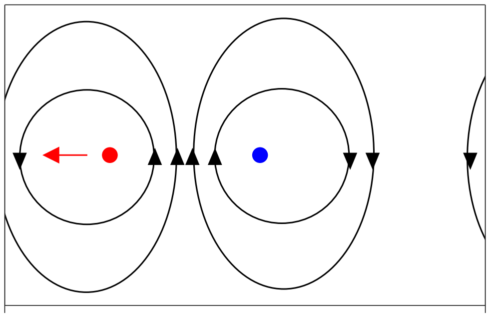

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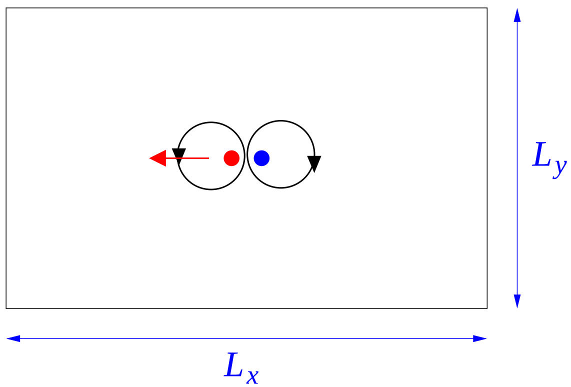

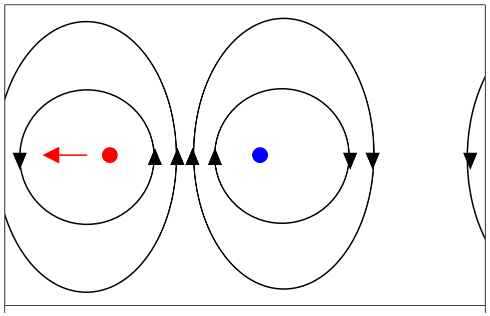

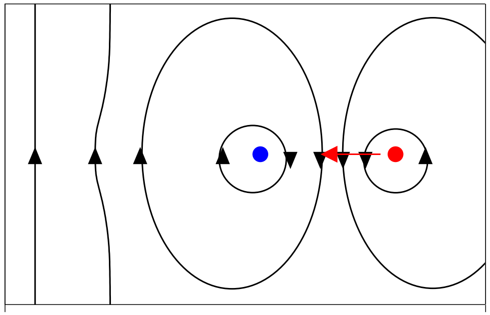

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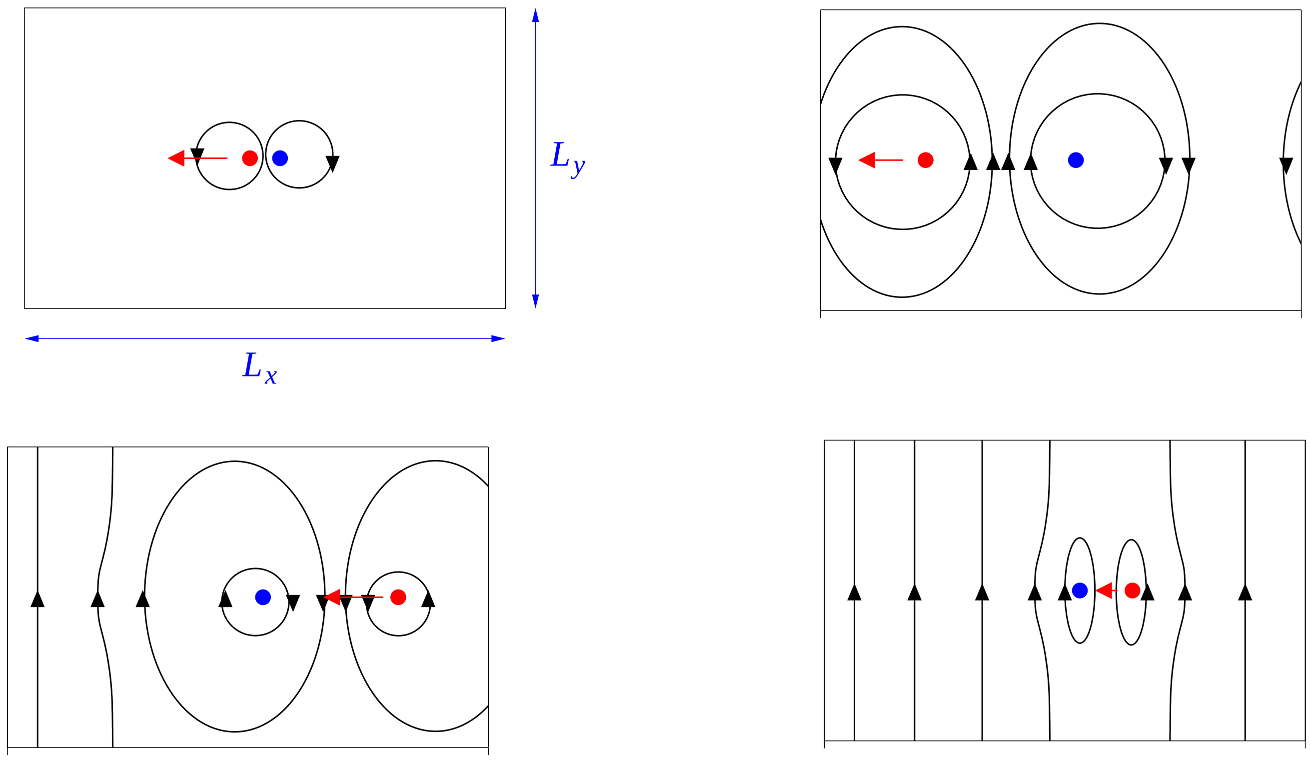
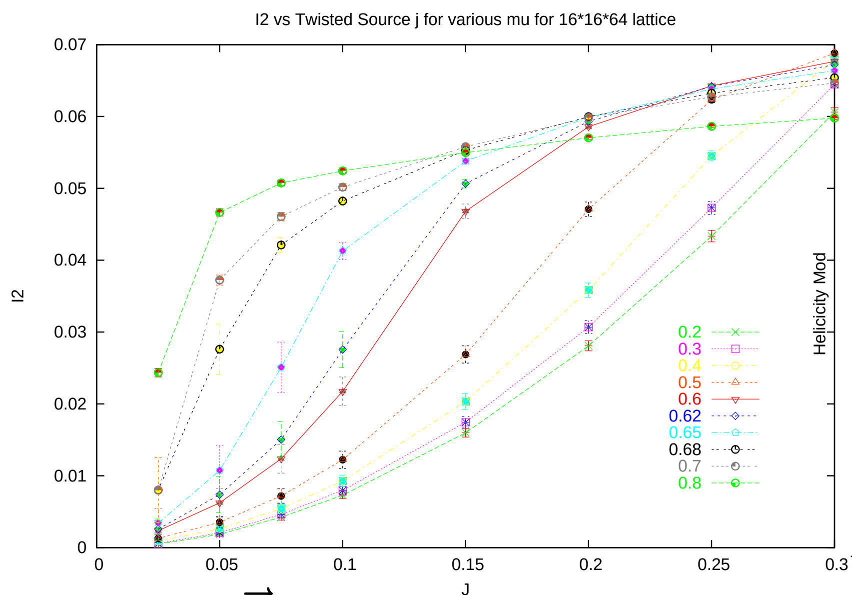
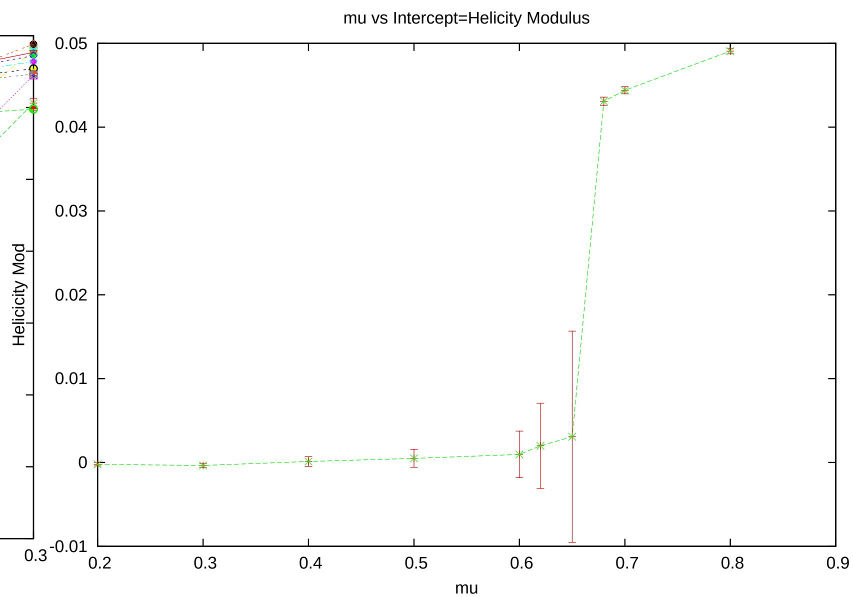


Vortex transport $\parallel \hat{x}$ induces a current density $J_{sy} = \frac{2\pi\Upsilon}{L_y}$

Energy required to change $\vec{J}_s \sim \ln L_x$

To test this scenario, with A. Sehra we are currently running simulations with a “twisted” source $j(x) = j_0 e^{i\theta(x)}$ with θ a periodic function of x.





Expect $\vec{J}_s = \langle \bar{\psi} \vec{\gamma} \psi \rangle = \frac{2\pi}{L} \Upsilon$ as $j_0 \to 0$

Initial results suggest helicity modulus Υ shows strong first-order transition as μ is increased

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Without a sign problem to worry about, simulations with $\mu \neq 0$ are in many respects easier than those with $T > 0$!

- Evidence for superfluidity in 3+1d
- Evidence for thin film superfluidity and new universality class in 2+1d
- For the future:
  is there a model with long-range interactions which interpolates between BEC and BCS?
  what is the physical origin of the sign problem?