Maximally Multipartite Entangled States and Statistical Mechanics

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Entanglement

Bipartite systems VS Multipartite systems

Two subsystems A and B: evaluate entanglement between them

Many subsystems: ?!?!

Entanglement

Consider two systems A and B in a state $|\eta\rangle$

If one can write the state in a factorized form $|\eta\rangle = |\psi\rangle_A |\phi\rangle_B$

then the state is SEPARABLE. Otherwise it is ENTANGLED.

Objective: characterizing ***Multipartite Entanglement***

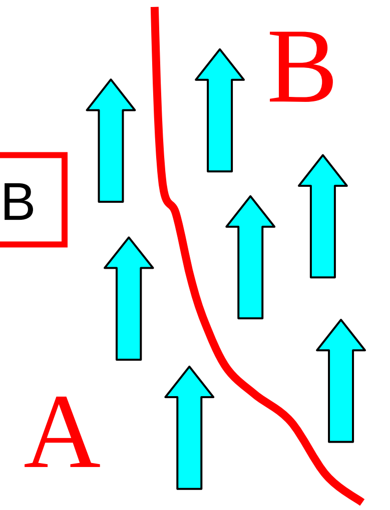
Objective: define ***Maximally Multipartite Entangled States***

Applications in many-body physics (see: Amico et al. Rev. Mod .Phys. 2008)

Entanglement

We consider an ensemble of ***n*** two-level systems (qubits) in the state

$$\rho = |\psi\rangle\langle\psi|$$



and a partition of the ensemble in two subsystems A and B

What is the bipartite entanglement between A and B?

For a generic state one can find its diagonal form

$$\rho = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \lambda_N \end{pmatrix}$$

N = dimension of the Hilbert space

λ_i = eigenvalues of ρ

$$L(\rho) = \mathrm{Tr}(\rho^2) = \sum_i \lambda_i^2$$

Purity (linear entropy): a measure of **bipartite** entanglement

Entanglement

The reduced density matrix of subsystem A is obtained by tracing on the degrees of freedom of B

$$\rho_A = \mathrm{tr}_B \rho$$

Its purity is

$$\pi_A = L(\rho_A) = \mathrm{tr}_A \rho_A^2$$

Max entangled for bipartition A-B

All eigenvalues = 1/N_A

$$\frac{1}{N_A} \leq \pi_A \leq 1$$

Separable for bipartition A-B

Only one eigenvalue different from 0

Dimension of the Hilbert space of A

Entanglement is “encoded” in the eigenvalues of the density matrix

$$|\eta\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B) \Rightarrow \rho_A = \mathrm{Tr}_\mathrm{B}(\rho) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \Rightarrow \pi_A = \frac{1}{2}$$

$$|\eta\rangle = |0\rangle_A|1\rangle_B \Rightarrow \rho_A = \mathrm{Tr}_\mathrm{B}(\rho) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \pi_A = 1$$

Characterization of multipartite entanglement

The quantity π_A completely defines the **BIPARTITE ENTANGLEMENT** (one number is sufficient). **It depends on the bipartition**.

What about **MULTIPARTITE ENTANGLEMENT**? The numbers needed to characterize the system scale exponentially with its size.

Seminals ideas from
Man'ko, Marmo, Sudarshan, Zaccaria:(J. Phys. A 02-03)
Parisi: complex systems

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Statistical methods

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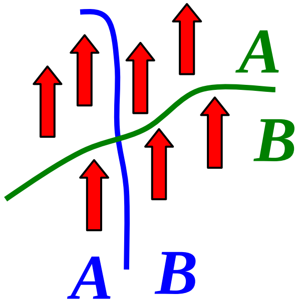
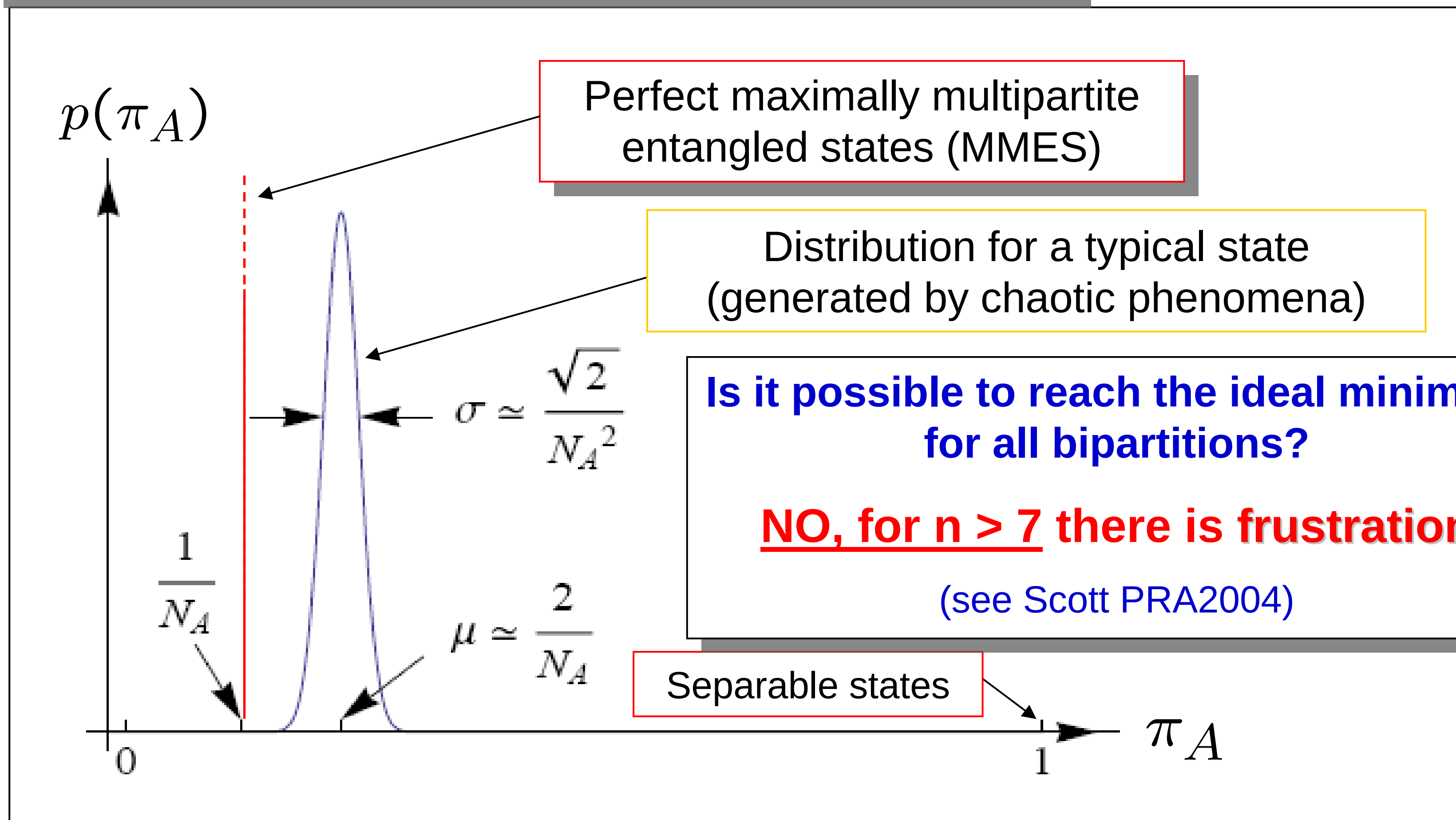
- The distribution of π_A characterizes the entanglement of the system.
- The **average** will be a measure of the amount of entanglement in the system, while the **variance** will measure how well such entanglement is distributed: a smaller variance will correspond to a larger insensitivity to the choice of the partition.
- (See Facchi, G.F., Pascazio [PRA 74, 042331 (2006)])

For chaotic systems



Chaotic phenomena can generate (typical) states with a large amount of entanglement (see Facchi, G.F. Pascazio, PRA 2006; Rossini, Benenti PRL 2008)



Is it possible to reach the ideal minimum for all bipartitions?

NO, for n > 7 there is frustration

(see Scott PRA2004)

Obtaining a MMES

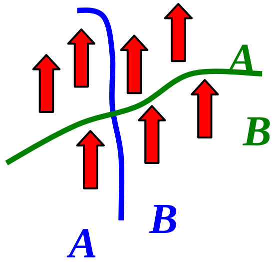
Maximally multipartite entangled state (**MMES**): minimizer of the *potential of multipartite entanglement* (see Facchi, G.F., Parisi, Pascazio PRA 2008)

$$\pi_{\mathrm{ME}} = \binom{n}{n_A}^{-1} \sum_{|A|=n_A} \pi_A$$

Minimization over balanced bipartitions

n_A is the number of qubits in subsystem A

Due to linearity, it inherits the bounds $1/N_A \leq \pi_{\mathrm{ME}} \leq 1$



If we want to reach a delta distribution…
we introduce the generalized cost function

$$\tilde{\pi}_{\mathrm{ME}}(\lambda) = \pi_{\mathrm{ME}} + \lambda \sigma_{\mathrm{ME}}$$

Lagrange multiplier > 0

$$\sigma^2_{\mathrm{ME}} = \binom{n}{n_A}^{-1} \sum_{|A|=n_A} (\pi_A - \pi_{\mathrm{ME}})^2$$

variance

Optimization

We search MMES of the form

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{i\varphi_k} |k\rangle$$

For a given bipartition we find

$$\pi_A = \frac{N_A + N_B - 1}{N} + \frac{1}{N^2} \sum_{\substack{l_A, l'_A \\ m_B, m'_B}} \cos\left(\varphi_{l_A m_B} - \varphi_{l'_A m_B} + \varphi_{l'_A m'_B} - \varphi_{l_A m'_B}\right)$$

2 qubits

$$\pi_{\mathrm{ME}}^{(2)} = \frac{3}{4} + \frac{1}{4}\cos(\varphi_0 - \varphi_1 - \varphi_2 + \varphi_3)$$

minimization →

$$\pi_{\mathrm{ME}}^{(2)} = \frac{1}{2}$$

3 qubits ⇓

$$\pi_{\mathrm{ME}}^{(3)} = \frac{1}{2}$$

4 qubits ⇓

$$\pi_{\mathrm{ME}}^{(4)} = \frac{1}{3} \neq \frac{1}{4}$$

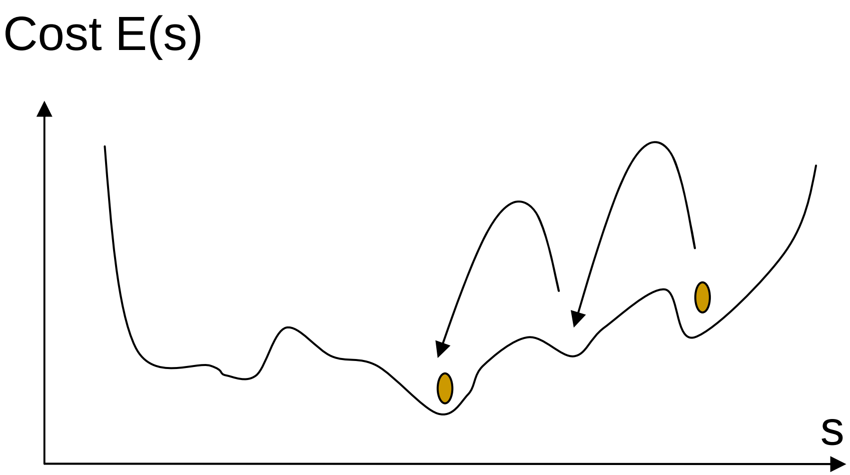
The system of 4 qubits is frustrated (we send to 0 the variance for weigths $\neq 1/\sqrt{N}$ but not a perfect MMES)

Larger systems

For larger system the optimization procedure is more difficult.

We find a number of local minima where deterministic algorithms get stuck

Stochastic algorithms → Simulated annealing (see Kirkpatrick et al., Science 1983)



Start in a configuration s. At each step the algorithm chooses a new configuration s' and probabilistically decides if let the system in s or move it to s'.

The acceptance probability must depend on the "energy difference" E(s')-E(s) and on a "temperature"; it is non zero when $\Delta E > 0$; thus it is possible to pass barriers.

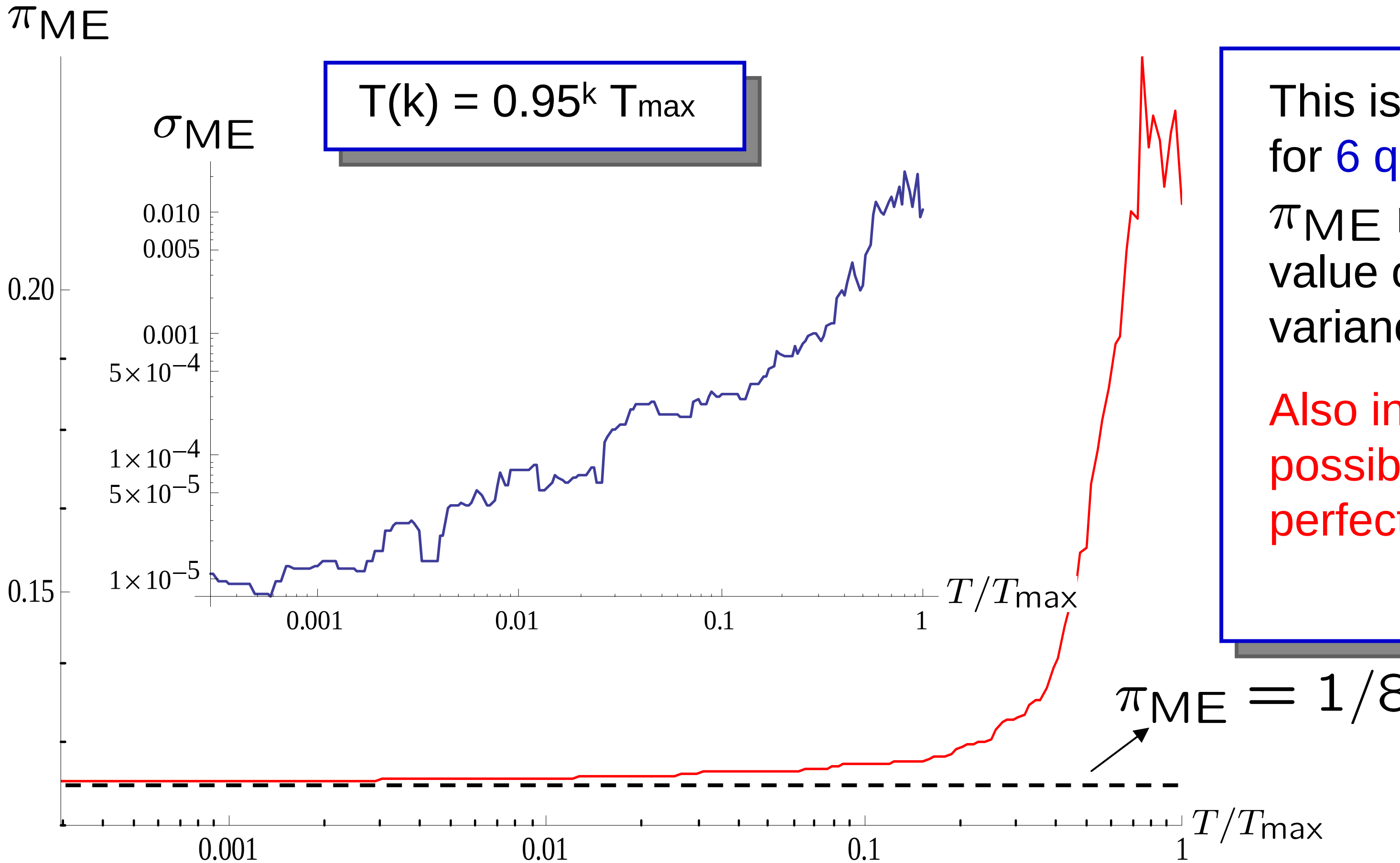
A simple choice is using the Metropolis algorithm with a Boltzmann factor.

The schedule for the temperature lowering depends on the problem (usually, the slower, the better).

5-6 qubits

For 5 qubits we tested the case of phases = 0 or π

It turns out that it is always possible to find a perfect MMES with these phases

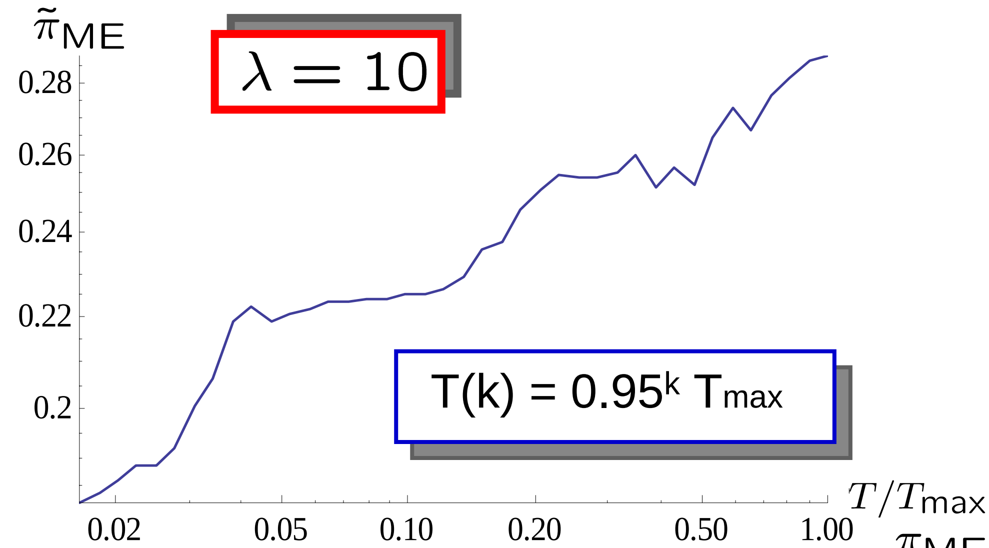


This is a typical run for 6 qubits.

π_{ME} reaches the value of 1/8 and the variance goes to 0.

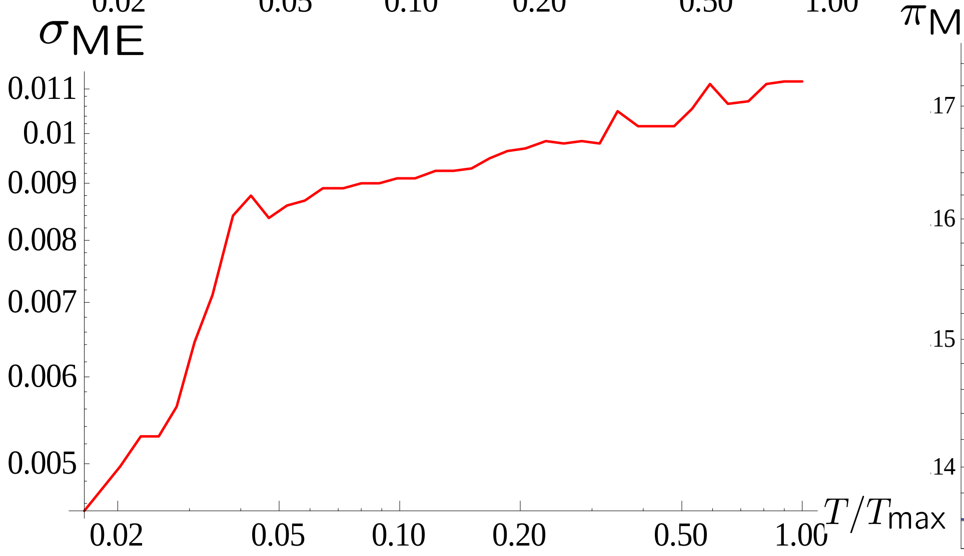
Also in this case it is possible to find a perfect MMES

7 qubits

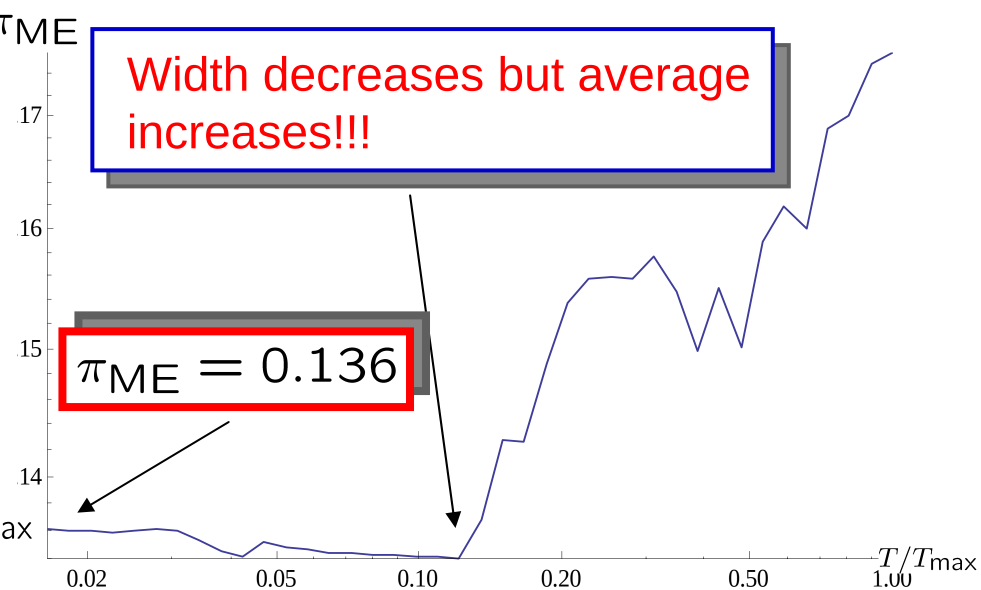


The numerical optimization of π_{ME} shows that it is not possible to reach the ideal value 1/8. Thus we have to optimize the cost function $\tilde{\pi}_{ME}$

In this case one has to find a compromise between average entanglement and width of the distribution.



Width decreases but average increases!!!



Let's change strategy…

The minimization problem becomes easily very complicated (system size + frustration). We search another strategy to define MMES…

We will recast the problem in a *classical statistical mechanical* problem.

We consider the state $|\psi\rangle = \sum_{k=1}^{N} z_k |k\rangle \qquad z_k \in \mathbf{C}$

The potential of entanglement is a function of the coefficients

$$\pi_{\mathrm{ME}} = \binom{n}{n_A}^{-1} \sum_{|A|=n_A} \pi_A = H(z)$$

The set of $z_k, \bar{z}_k$

Let's change strategy… (see Facchi, GF, Marzolino, Parisi, Pascazio arxiv:0803.4498)

We introduce the partition function:

$$Z(\beta) = \int d\mu_C(z) e^{-\beta H(z)}$$

with the measure $d\mu_C(z) = \prod_k dz_k d\bar{z}_k \delta\left(1 - \sum_k |z_k|^2\right)$ → normalization

and a fictitious inverse temperature β

A brief summary

$\beta \to +\infty$	$H = E_0$ (min)	MMES
$\beta \to 0$	$H \simeq \mu$	typical states
$\beta \to -\infty$	$H = 1$ (max)	separable states

$\mu \simeq 2/\sqrt{N}$

Statistical Mechanics

Suppose we have the distribution at infinite temperature $P_0(E)$

The distribution at ARBITRARY temperature is

$$P_\beta(E) = \frac{e^{-\beta E} P_0(E)}{\int_{E_0}^{1} dE e^{-\beta E} P_0(E)}$$

$$1/N_A \leq E_0(N_A) \leq \mu \leq 2/N_A \quad \lim_{N_A \to \infty} E_0(N_A) = 0$$

Limits for the distribution

$$P_{-\infty}(E) = \delta(E-1), \qquad P_{+\infty}(E) = \delta(E - E_0)$$

Statistical Mechanics

For the average we find

$$\langle H\rangle_\beta = \frac{1}{Z(\beta)}\int d\mu_C(z) H e^{-\beta H} = \int_{E_0}^{1} dE\,E P_\beta(E) = -\frac{\partial}{\partial\beta}\ln Z(\beta)$$

Limits:

$$\langle H\rangle_{-\infty} = 1, \qquad \langle H\rangle_{+\infty} = E_0$$

Derivative:

$$\frac{\partial}{\partial\beta}\langle H\rangle_\beta = -\langle H^2\rangle_\beta + \langle H\rangle_\beta^2 \equiv -\triangle H_\beta^2 \leq 0$$

Statistical Mechanics

We can evaluate the cumulants of the distribution at high temperature:

The average is the same of the purity $\mu_H \simeq 2/\sqrt{N}$

Variance: $\bar{\sigma}^2 = \triangle H_0^2 = \kappa_0^{(2)}(H)$

$$\bar{\sigma}^2 \sim 3\sqrt{2} N^{-4+\log_2 3} \simeq O(N^{-2.42})$$

For independent bipartitions: $\bar{\sigma}^2 \sim \sigma^2/N = O(N^{-3})$

There is an interaction among the bipartitions

Gaussian Approximation

Higher order cumulants decrease faster ⇒ Gaussian approximation

$$P_0(E) \sim \frac{1}{\sqrt{2\pi\bar{\sigma}^2}} \exp\left(-\frac{(E-\mu)^2}{2\bar{\sigma}^2}\right)$$

⇓

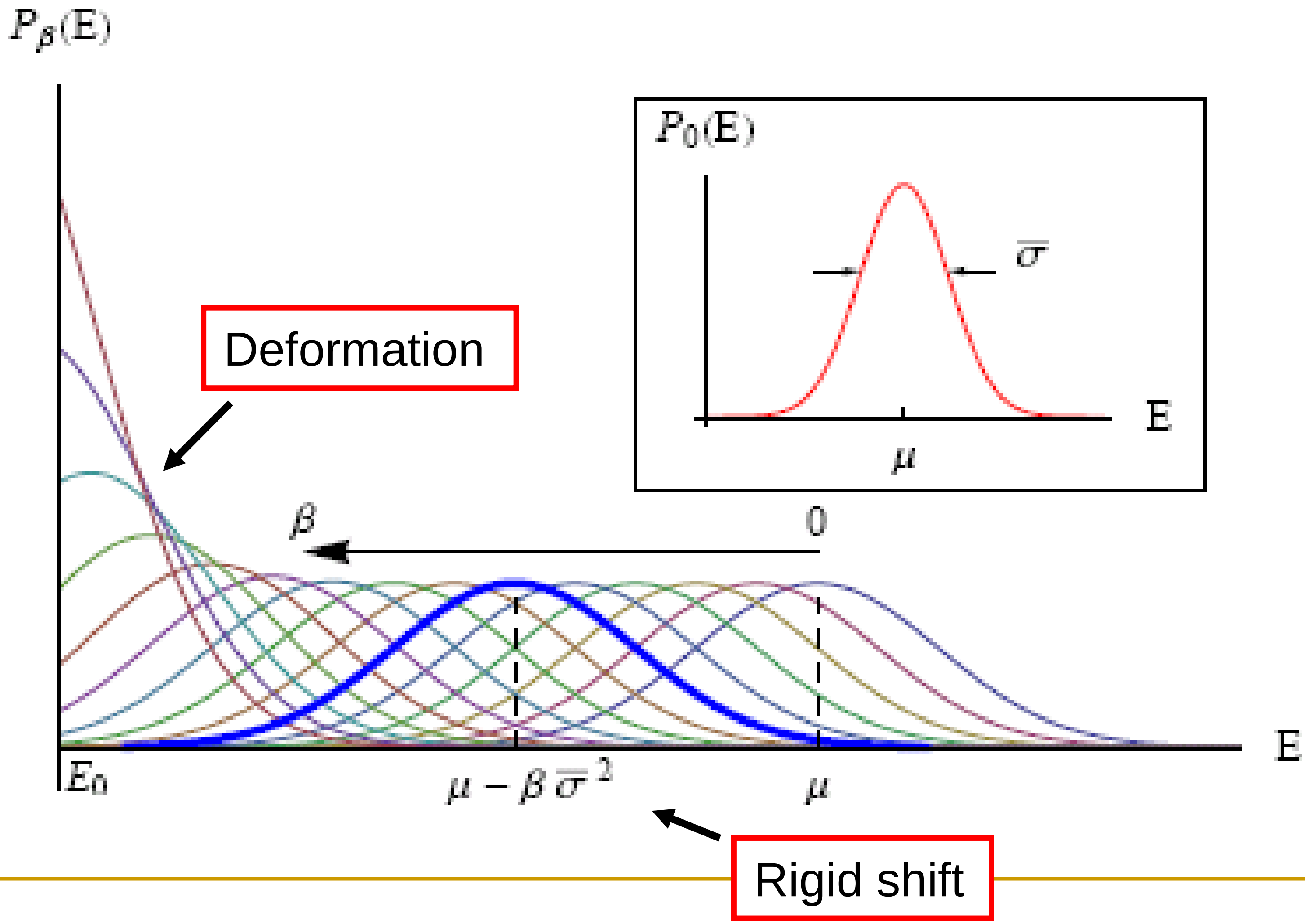
$$P_\beta(E) \sim \frac{1}{\sqrt{2\pi\bar{\sigma}^2}} \exp\left(-\frac{\left(E-\mu+\beta\bar{\sigma}^2\right)^2}{2\bar{\sigma}^2}\right)$$

Valid if $\mu - \beta\bar{\sigma}^2 - \bar{\sigma} > 0$

$$\beta < \mu/\bar{\sigma}^2 = O(N^{7/2-\log_2 3}) \simeq O(N^{1.92})$$

Dependence on the "Temperature"

When the tail "touches" the minimum, the distribution is deformed



Dependence on the "Temperature"

This picture is valid if $P_0(E_0) \neq 0$

What if $P_0(E_0) = 0$?

$$P_\beta(E) \sim \frac{\beta^{\ell+1}}{\ell!}(E - E_0)^\ell e^{-\beta(E-E_0)}$$

order of the first non vanishing derivative

Conclusions

We defined a characterization of multipartite entanglement that is based on the framework of bipartite entanglement but with statistical information.

We defined an optimization problem for deriving a class of Maximally Multipartite Entangled States (MMES)

We recasted the problem in terms of a classical statistical mechanical problem

We obtained a non trivial form of the second cumulant of the energy distribution and some features of the high temperature behaviour.