Fluctuation relations in non-equilibrium stationary states Ising models

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Outline

- Introduction: generality about the FR
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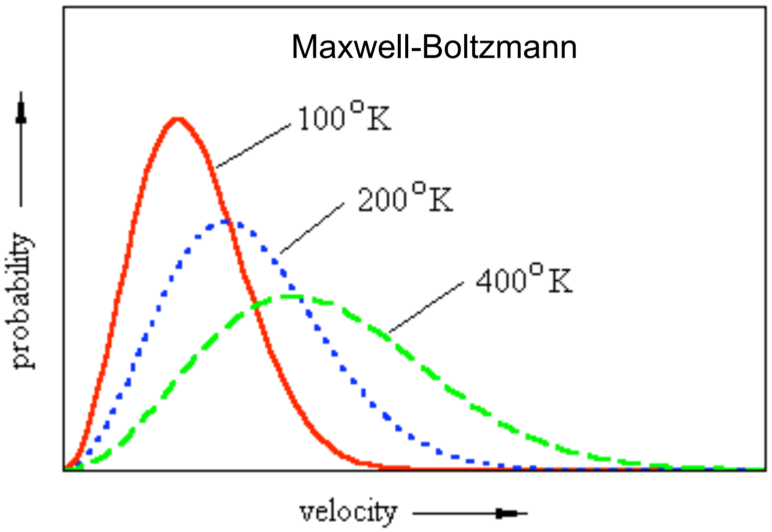
Fluctuations in non equilibrium systems



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External driving or thermal gradients

Motivations

- Is the FR realized in popular many degree of freedom statistical models (e.g. Ising model)?
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Heat and Work definition

DRIVING

$$Q[x] = \sum_{t=0}^{\tau-1} [E(t)_{x(t+1)} - E(t)_{x(t)}]$$

$$W[x] = \sum_{t=0}^{\tau-1} [E(t+1)_{x(t+1)} - E(t)_{x(t+1)}]$$

$$\Delta E = E(\tau)_{x(\tau)} - E(0)_{x(0)} = W + Q$$

Work moves counter

Heat moves counter

TWO TEMPERATURES

$$Q_i[x] = \sum_{t=0}^{\tau-1} [E(t)_{x(t+1)} - E(t)_{x(t)}]|_i$$

(where are considered only the spin-flip of spin in contact with the bath at temperature i)

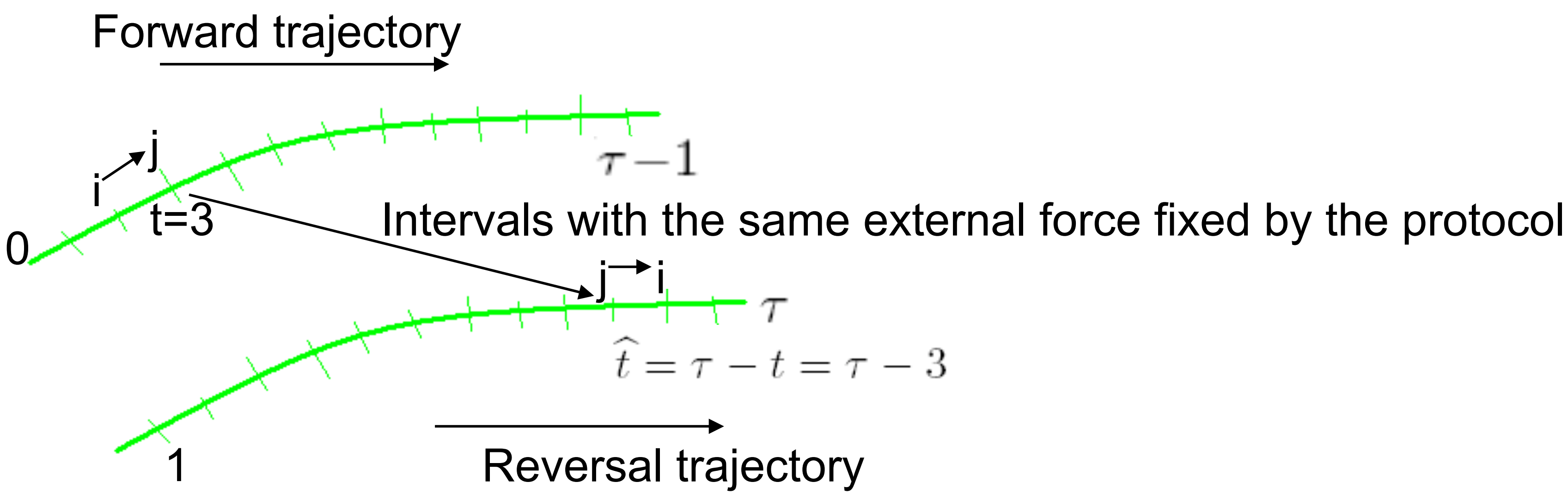
Microscopic reversibility: Markov chains

Equilibrium:

$$\pi_{ij}\lambda_j = \pi_{ji}\lambda_i \qquad \lambda_i = e^{-\beta(E_i - F)}$$

External work:

Non homogeneous dynamics $\pi = \pi(t)$ because of the time dependent protocol of the externl force



λ is the invariant distribution of $\pi(t)$ for every t
i.e. the unperturbed dynamics preserves the equilibrium distribution

Transition matrix of the reversal path

$$\pi_{ij}(\tau - t)\lambda_j(\tau - t)_{x(\tau-t)} = \widehat{\pi}_{ji}(t)\lambda_i(\tau - t)_{x(\tau-t+1)}$$

$$\frac{P(traj)}{P(-traj)} = \frac{P[x(0), \pi]}{P[\widehat{x}(0), \widehat{\pi}]} =$$

$$= \frac{\pi_{x(1)x(0)} \cdots \pi_{x(\tau)x(\tau-1)}}{\lambda^{-1}_{x(\tau)}(\tau-1)\pi_{x(\tau)x(\tau-1)}(\tau-1)\lambda_{x(\tau-1)}(\tau-1)\ldots\ldots\lambda^{-1}_{x(1)}(0)\pi_{x(1)x(0)}(0)\lambda_{x(0)}(0)} =$$

$$= \prod_{t=0}^{\tau-1}\frac{\lambda_{x(t+1)}(t)}{\lambda_{x(t)}(t)} = e^{-\beta\sum_{t=1}^{\tau-1}[E(t)_{x(t+1)} - E(t)_{x(t)}]} = e^{-\beta Q}$$

$$\boxed{\frac{P[traj]}{P[-traj]} = e^{-\beta Q[x]}}$$

Microscopic reversibility

The FR for systems in contact with two heat baths

Equilibrium: $\pi_{ij}\lambda_j = \pi_{ji}\lambda_i \longrightarrow e^{\frac{q}{T}}\pi_{ij}^{q} = \pi_{ji}^{-q}$ $\quad q = E_i - E_j$

$\lambda_i = e^{-\beta(E_i - F)}$

Two temperatures: $e^{\frac{q_1}{T_1}+\frac{q_2}{T_2}}\pi_{ij}^{q_1,q_2} = \pi_{ji}^{-q_1,-q_2}$ Generalized detailed balance

$$\frac{Prob(traj)}{Prob(-traj)} = \frac{\prod_{i=1}^{\tau-1}\pi^{q_{1,i},q_{2,i}}(C_{i+1},C_i)\prod_{i=1}^{\tau-1}P_{H,i}}{\prod_{i=1}^{\tau-1}\pi^{-q_{1,i},-q_{2,i}}(C_i,C_{i+1})\prod_{i=1}^{\tau-1}P_{H,i}} =$$

$$e^{-\sum_{i=1}^{\tau-1}[\frac{q_{1,i}}{T_1}+\frac{q_{2,i}}{T_2}]} = e^{-\frac{Q_1}{T_1}-\frac{Q_2}{T_2}} = e^{-\frac{Q_1}{T_1}-\frac{\Delta E - Q_1}{T_2}} = e^{-Q_1(\frac{1}{T_1}-\frac{1}{T_2})-\frac{\Delta E}{T_2}}$$

. . . .

$\tau \to \infty \Rightarrow Q_1 >> \Delta E$ and $\sum_{traj|Q_1} \longrightarrow$ $\boxed{\frac{P(Q_1)}{P(-Q_1)} = e^{-(\beta_1-\beta_2)Q_1}}$

Two models: static and dynamic

Transition rates: $$\pi(C',C) = \begin{cases} k e^{-\beta_i \Delta E(C',C)} & \Delta E(C',C) > 0 \\ k & \text{otherwise} \end{cases}$$

$$k \ni' \sum_{C'} \pi(C',C) = 1$$

Single spin-flip & Kawasaki

$$i = \begin{cases} 1 & 1 < x \leqslant \frac{L}{2} \\ 2 & \frac{L}{2} < x \leqslant L \end{cases}$$

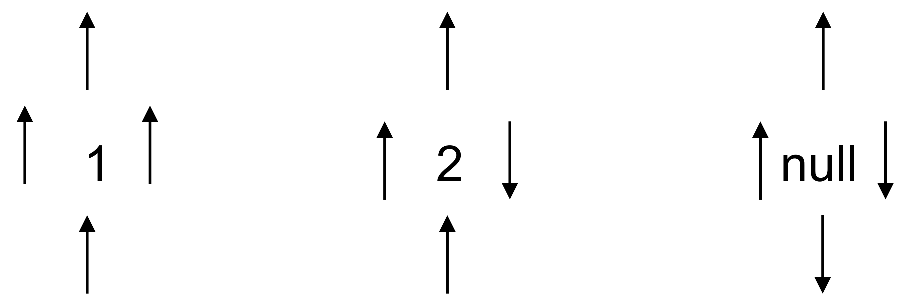
- Single spin-flip dynamics
- Kawasaki dynamics: spin-flip of randomly chosen pairs of first neighbour opposite spins

Dynamic model

$$i = \begin{cases} 2 & \frac{1}{2}\left|\sum_{\langle j \rangle i} \sigma_i\right| = 1 \\ 1 & \frac{1}{2}\left|\sum_{\langle j \rangle i} \sigma_i\right| = 2 \end{cases}$$

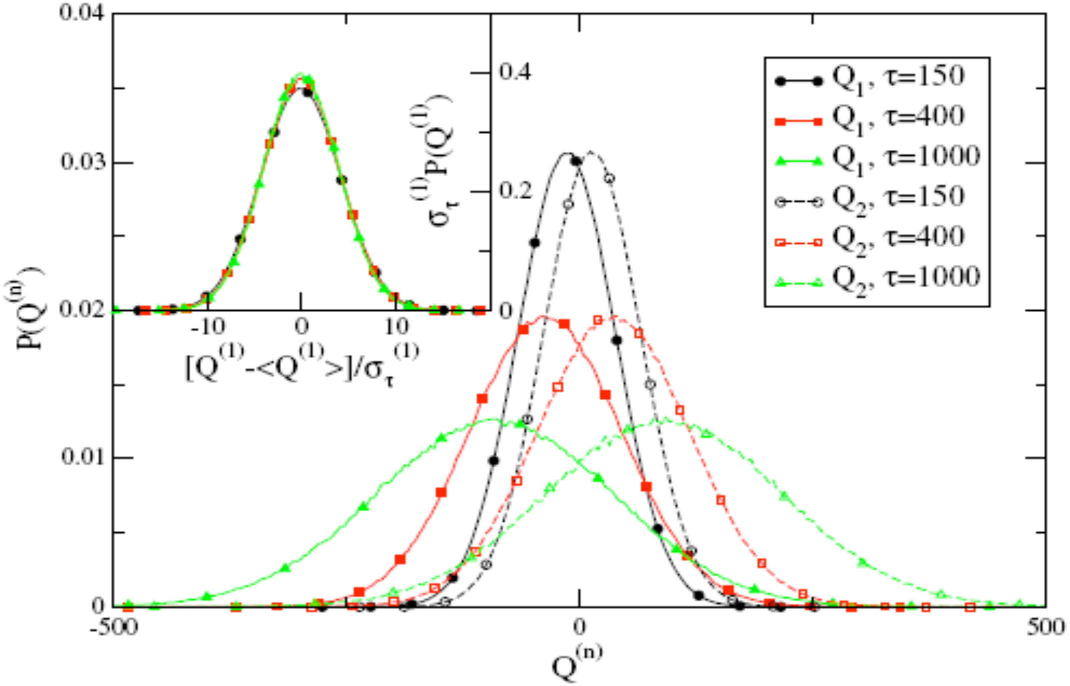
with $\beta_2 < \beta_1$

- Dynamic (Mendez et al.) model

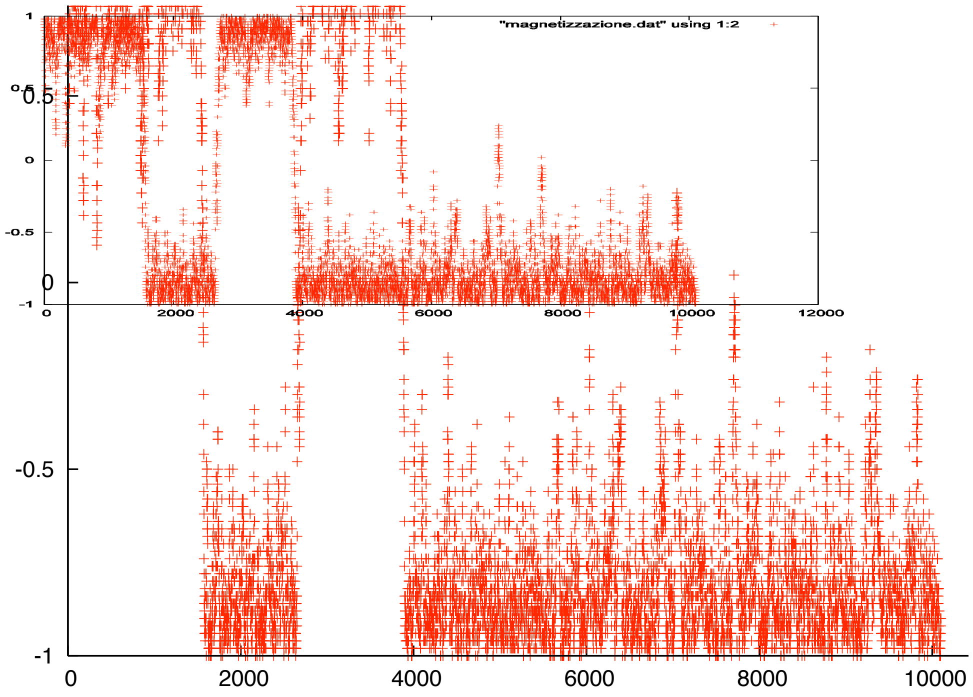


Ising model distribution

Above Tc: T_1=2.9 and T_2=3, L=10



Magnetization jumps



Fluctuation-dissipation limit

$$T_2 - T_1 \to 0^+$$

$\frac{\langle Q_1^2 \rangle_{eq}}{\tau} \to D$ for $T_1 = T_2 \longrightarrow D$ is the fluctuation coefficient

$\frac{\langle Q_1 \rangle}{\tau} \to (T_1 - T_2) j$ for $T_2 - T_1 \to 0^+ \longrightarrow j$ is the linear response

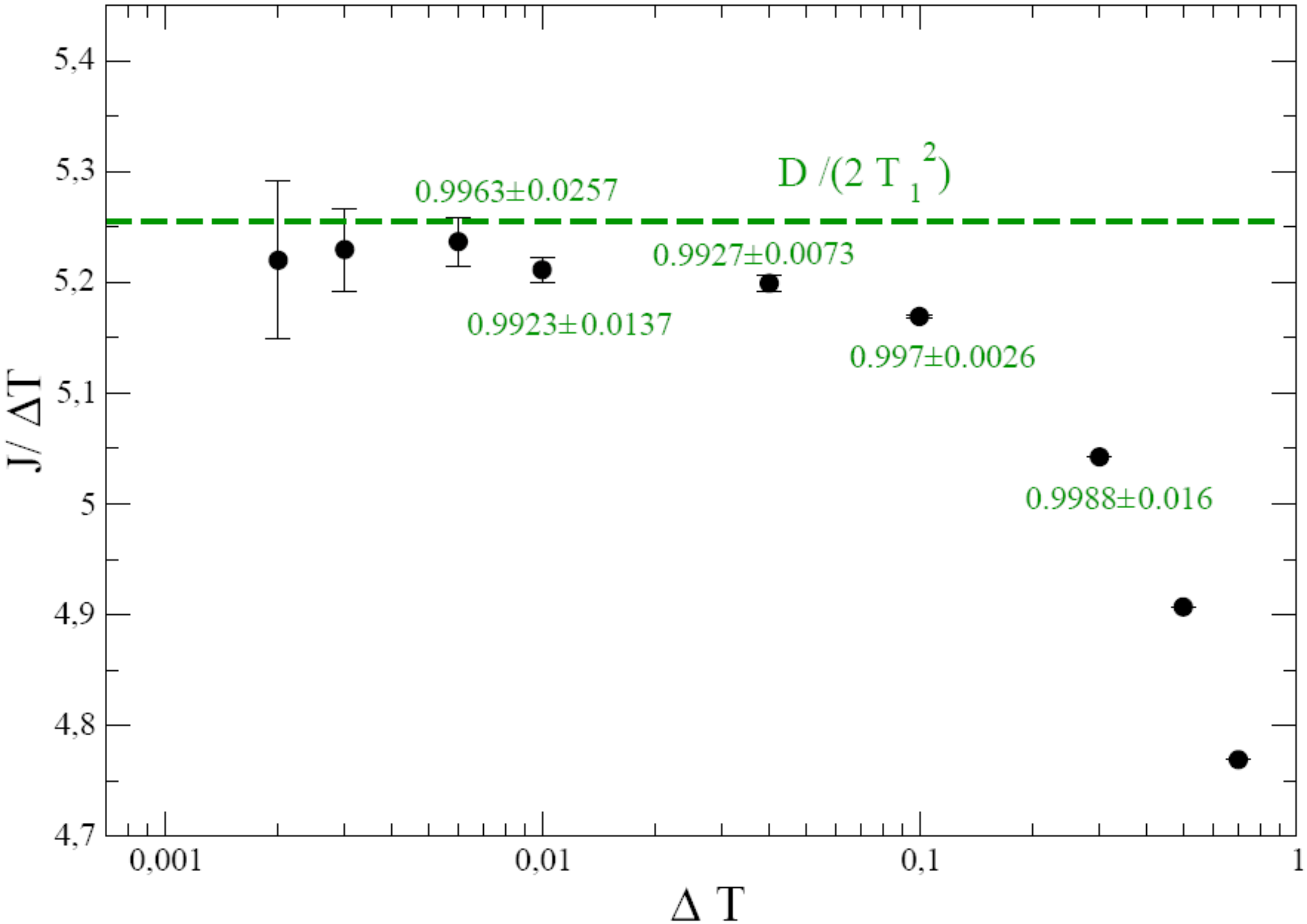
With these definition and exploiting the FR for gaussian fluctuations:

$$\lim_{T_2 - T_1 \to 0^+} \frac{\mu_1}{\tau (T_1 - T_2)} = \frac{\sigma_{eq}^2}{2 T_1^2 \tau} = \frac{D}{2 T_1^2}$$

$$\frac{j}{\Delta T} = \frac{D}{2 T_1^2}$$

Einstein relation for a two temperature system

Fluctuation-dissipation limit



Corrections and scaling

Defining ϵ as the correction to the slope 1 (the FR) recovered for $\tau \to \infty$

$$\frac{P(Q_1)}{P(-Q_1)} = e^{-Q_1(\beta_1-\beta_2)(1-\epsilon)}$$

$P(Q_1)$ gaussian + $\frac{\mathcal{P}^{staz}(\tau)}{\mathcal{P}^{staz}(0)}$ One free parameter includes the form of this ratio, which contains the interaction between the two bulk at different temperatures

CORRECTIONS

$$\epsilon^{(n)}\Delta\beta^{(n')} \simeq -\frac{\langle Q^{(n)}(\tau)\rangle}{(\sigma_\tau^{(n)})^2} \mp \sqrt{\left(\frac{\langle Q^{(n)}(\tau)\rangle}{(\sigma_\tau^{(n)})^2}\right)^2 + \frac{v_{\Delta E}^2}{(\sigma_\tau^{(n)})^2}(\Delta\beta^{(n')})^2}$$

SCALING

$$\lim_{\tau\to\infty} \epsilon^{(n)} = \frac{\Delta\beta^{(n')} v_{\Delta E}^2 (\sigma_\tau^n)^2}{2\langle Q^{(n)}(\tau)\rangle^2} \sim \frac{v_{\Delta E}^2}{(\sigma_\tau^n)^2} \sim \frac{1}{\tau}$$

Corrections trend with time and size

	L	τ
$v^2_{\Delta E}$	L^2	constant
σ^2_τ	static L dynamic L^2	τ

$$v^2_{\Delta E} \sim \sigma^2_{\Delta E} \sim L^2$$

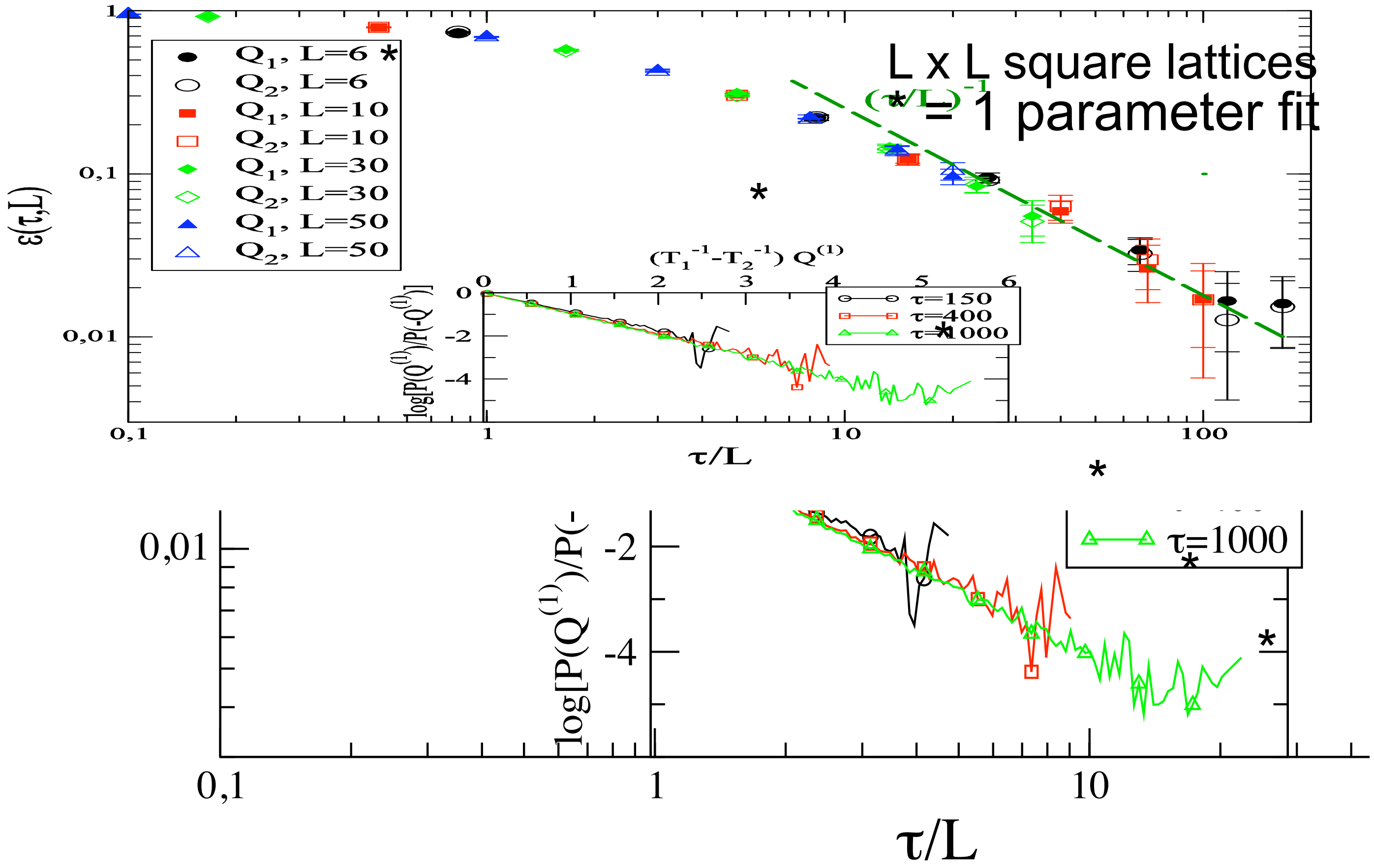
$$FR_{\tau\to\infty} + gaussianity : \frac{\langle Q_\tau \rangle}{\sigma^2_\tau} = -\frac{\beta_1 - \beta_2}{2} = constant\ with\ L\ and\ \tau$$

$$\sigma^2_\tau \sim_{L,\tau} \langle Q_\tau \rangle \sim\ number\ of\ interface\ links\ \propto \begin{cases} \text{static} & L \\ \text{dynamic} & L^2 \end{cases}$$

$$\epsilon(\tau, L) \sim_{\tau\to\infty} \frac{v^2_{\Delta E}}{\sigma^2_\tau} \sim \begin{cases} \text{static} & \frac{L^2}{L\tau} = \frac{L}{\tau} \\ \text{dynamic} & \frac{L^2}{L^2\tau} = \frac{1}{\tau} \end{cases}$$

Scaling Ising above Tc

Above Tc: T_1=2.9 and T_2=3

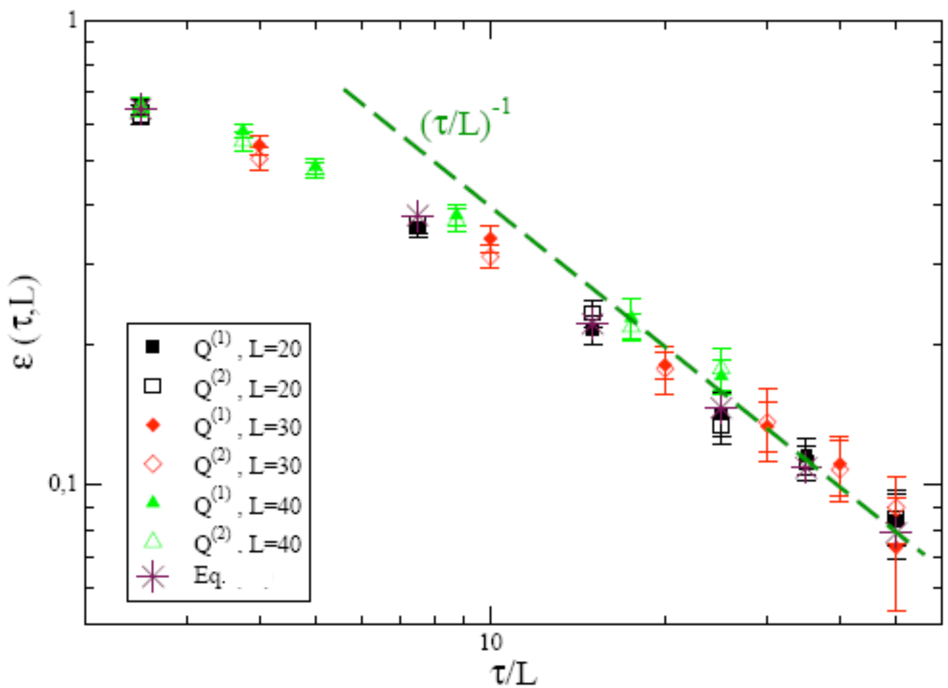


$$\epsilon(\tau, L) = 1 - slope$$

$$\epsilon(\tau, L) = f(\tau/L) \sim 1/x$$

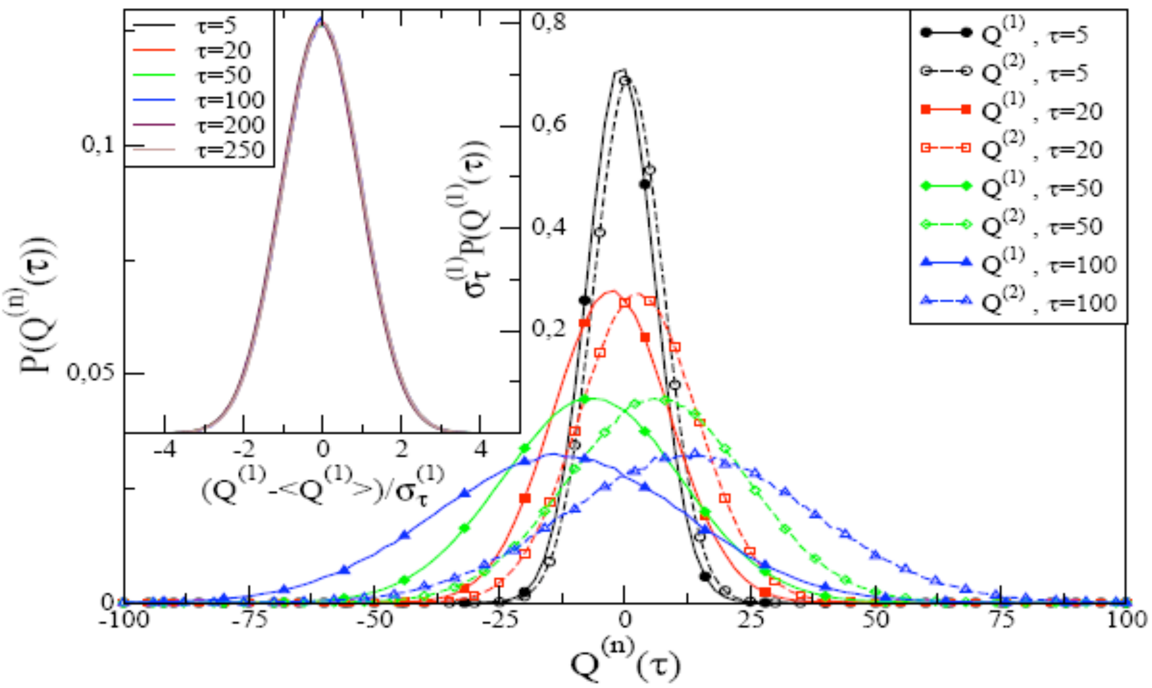
Kawasaki corrections and scaling

Above Tc: T_1=2.9 and T_2=3

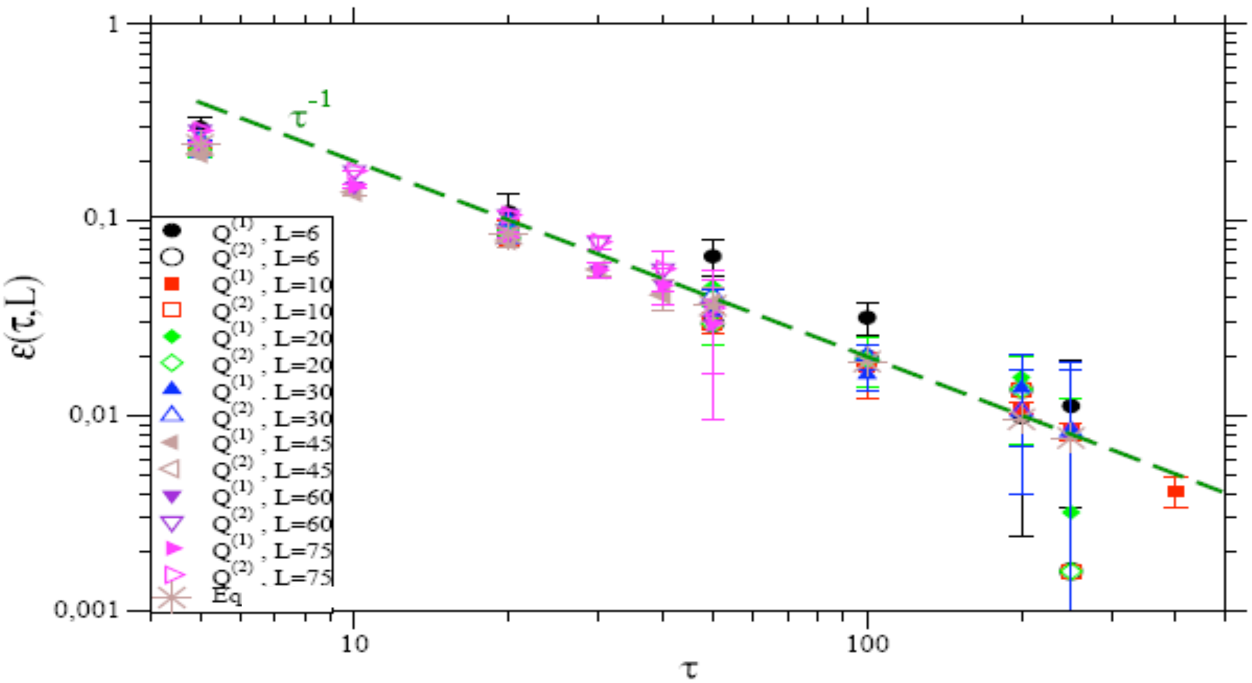


Dynamic model distributions

Above Tc: T_1=3. and T_2=3.1, L=10

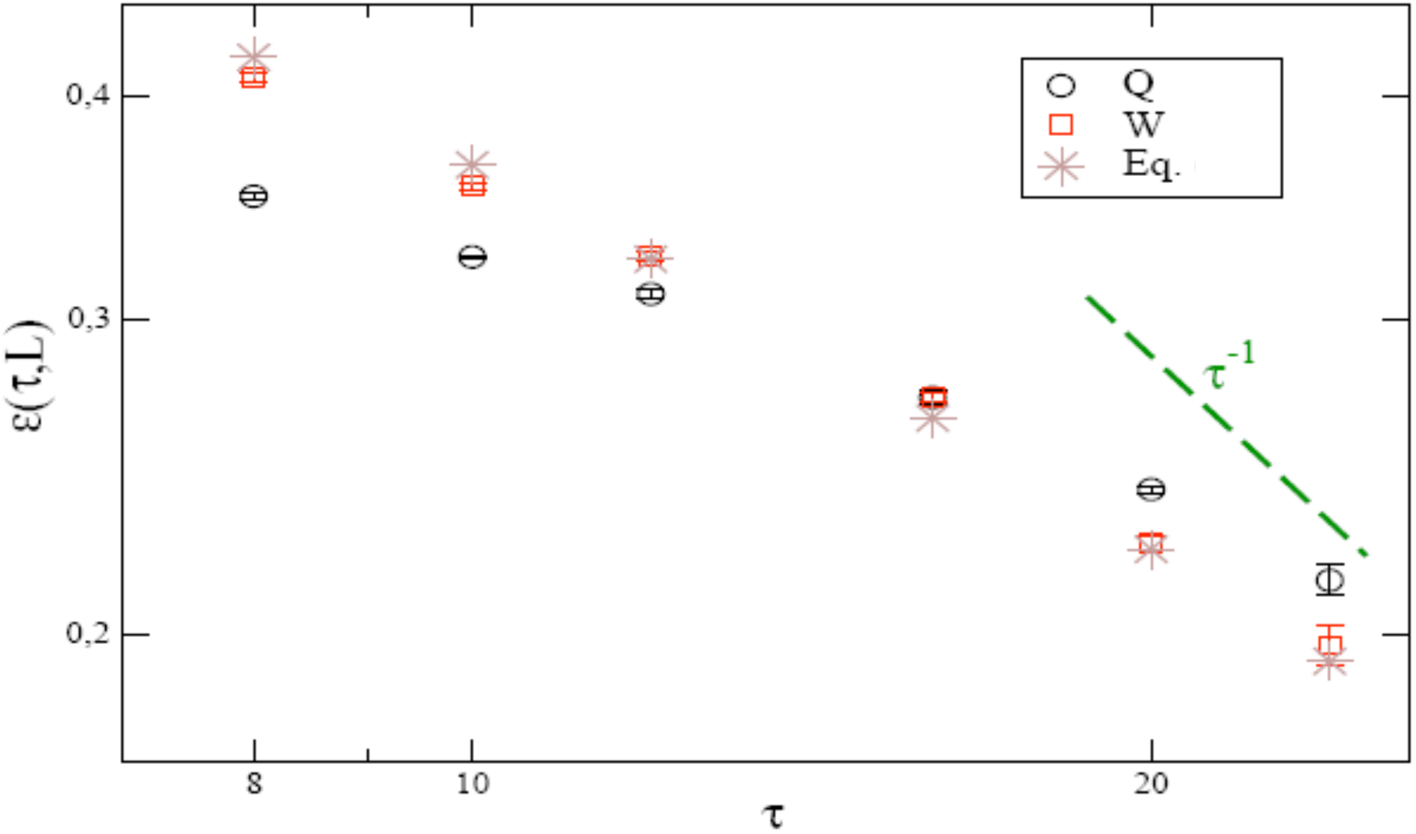


Dynamic model corrections and scaling



Above Tc: T_1=3. and T_2=3.1

Work corrections and scaling in an Ising system with shear



L=10

$$\epsilon^{(\mathcal{O})} = \mp \frac{T\langle \mathcal{O}(\tau)\rangle}{(\sigma_\tau^{(\mathcal{O})})^2} + T\sqrt{\left(\frac{\langle \mathcal{O}(\tau)\rangle}{(\sigma_\tau^{(\mathcal{O})})^2}\right)^2 + \frac{v^2}{(\sigma_\tau^{(\mathcal{O})})^2}}$$

- for $\mathcal{O} = \mathcal{W}$

+ for $\mathcal{O} = \mathcal{Q}$.

Conclusions

- The FR is realized also below T_c when the PDFs are not gaussian (in a way independent by the linear response theory)
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Thank you!