The QCD critical line from the method of analytic continuation

A. Papa

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References

• P. Cea, L. Cosmai, M. D'Elia, A.P., Phys. Rev. D77 (2008) 051501 [arXiv:0712.3755]
• P. Cea, L. Cosmai, M. D'Elia, C. Manneschi, A.P., Phys. Rev. D80 (2009) 034501 [arXiv:0905.1292]
• P. Cea, L. Cosmai, M. D'Elia, A.P., Phys. Rev. D81 (2010) 094502 [arXiv:1004.0184]
• P. Cea, L. Cosmai, M. D'Elia, A.P., F. Sanfilippo, in preparation

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Outline

1. Introduction
   - QCD phase diagram
   - QCD with non-zero baryon density and the sign problem
   - The method of analytic continuation

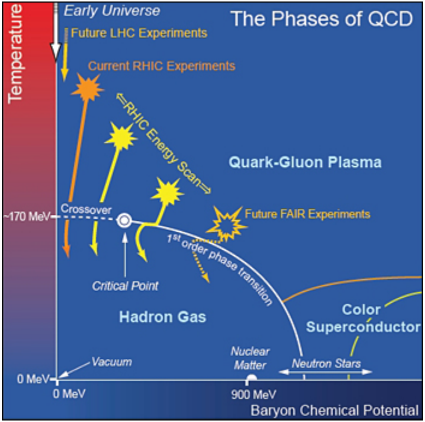
2. Investigations in QCD-like theories free of the sign problem
   - Two-color QCD with $n_f = 8$
   - Finite isospin SU(3) with $n_f = 8$

3. Application to QCD with $n_f = 4$ and $n_f = 2$
   - SU(3) with $n_f = 4$
   - SU(3) with $n_f = 2$ **(new)**
   - Finite isospin SU(3) with $n_f = 2$ **(new)**

4. Conclusions

Outline

QCD phase diagram



(from bnl.gov)

Important implications in heavy ion collisions, in cosmology and in physics of compact stars.

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QCD at non-zero temperature and density

- Lattice is the main non-perturbative tool for the investigation of the QCD phase diagram

- Non-zero temperature:  $T = \frac{1}{N_\tau a(\beta)}$, $\beta = \frac{2N}{g^2}$

- Non-zero density:  sign problem!

  Importance sampling requires positive weights, but in

$$Z(T,\mu) = \int [dU]\, e^{-S_G[U]} \det[M(\mu)]$$

  the fermionic determinant $\det[M(\mu)]$ is complex for $\mu \neq 0$ in SU(3).

  Exceptions:
  - imaginary chemical potential: $\mu = i\mu_I$
  - SU(2) or two-color QCD
  - isospin chemical potential: $\mu_u = -\mu_d$

Outline

The method of analytic continuation

- Perform Monte Carlo numerical simulations at some selected imaginary values of the chemical potential, $\mu = i\mu_I$, thus getting data points with their statistical uncertainties
- Interpolate the results obtained by a suitable function of μ_I^2
- Analytically continue to real chemical potentials: $\mu_I \to -i\mu$

A bit of history:

- idea of formulating a theory at imaginary chemical potential [M.G. Alford, A. Kapustin, F. Wilczek, 1999]
- test of effectiveness in strong-coupling QCD [M.P. Lombardo, 2000]
- thereafter, a lot of applications to QCD and tests in QCD-like theories and in spin models

- Applications in QCD:
  - $n_f = 2$ staggered [Ph. de Forcrand, O. Philipsen, 2002] [M. D'Elia, F. Sanfilippo, 2009]
  - $n_f = 3$ staggered [Ph. de Forcrand, O. Philipsen, 2003]
  - $n_f = 4$ staggered [M. D'Elia, M.P. Lombardo, 2003-2004] [V. Azcoiti *et al.*, 2004-2005] [M. D'Elia, F. Di Renzo, M.P. Lombardo, 2007]
  - $n_f = 2 + 1$ staggered [Ph. de Forcrand, O. Philipsen, 2007]
  - $n_f = 2$ Wilson [L.-K. Wu, X.-Q. Luo, H.-S. Chen, 2007] [A. Nagata, K. Nakamura, 2011]
  - $n_f = 4$ Wilson [H.-S. Chen, X.-Q. Luo, 2005]

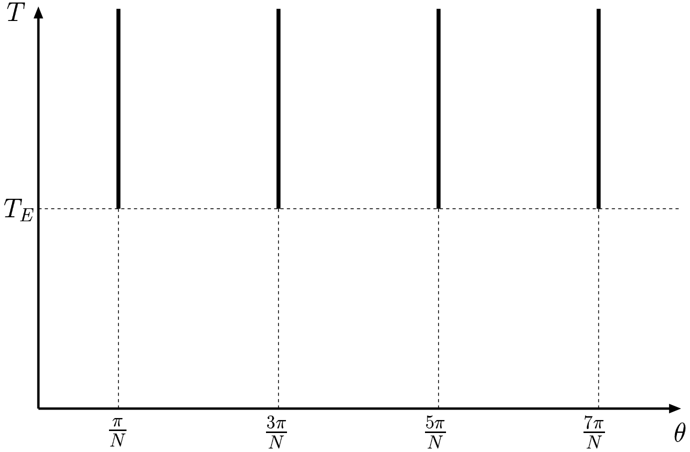
- Tests:
  - 3*d* SU(3) + adj. Higgs [A. Hart, M. Laine, O. Philipsen, 2001]
  - SU(2), $n_f = 8$ staggered [P. Giudice, A.P., 2004]
  - SU(3), $n_f = 8$ staggered [S. Conradi, M. D'Elia, 2007]
  - SU(2) via chiral RMT model [Y. Shinno, H. Yoneyama, 2009]
  - 3*d* 3-state Potts model [S. Kim *et al.*, 2005]
  - 2*d* Gross-Neveu at large *N* [F. Karbstein, M. Thies, 2006]

Drawbacks

1. a practical one: Monte Carlo simulations yield data points with statistical uncertainties at fixed values of the imaginary chemical potential; the interpolation of these points is not unambiguous
2. a principle one: the theory at imaginary chemical potential has its own non-analyticities and is periodic in the variable $\theta = \mu_I / T$ (period $2\pi / N$) [A. Roberge, N. Weiss, 1986]

$\Rightarrow$ the region effectively available for Monte Carlo simulations is limited by the condition $\mu_I / T \lesssim 1$

- The combination of these two drawbacks implies that the analytic continuation is expected to work for real chemical potentials satisfying $\mu_R / T \lesssim 1$.



T
T_E
$\frac{\pi}{N}$
$\frac{3\pi}{N}$
$\frac{5\pi}{N}$
$\frac{7\pi}{N}$
θ

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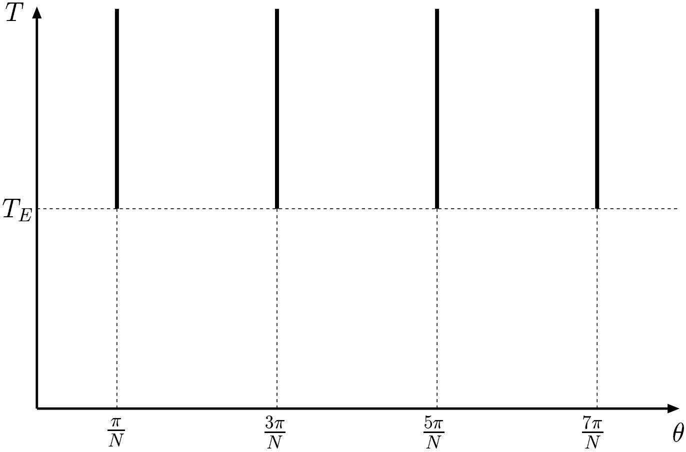
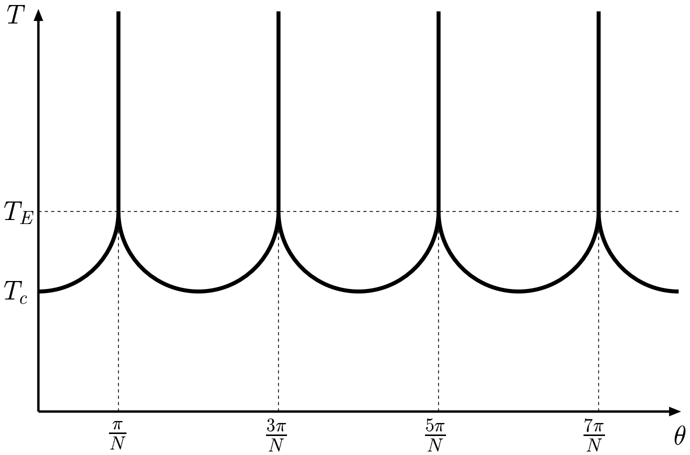
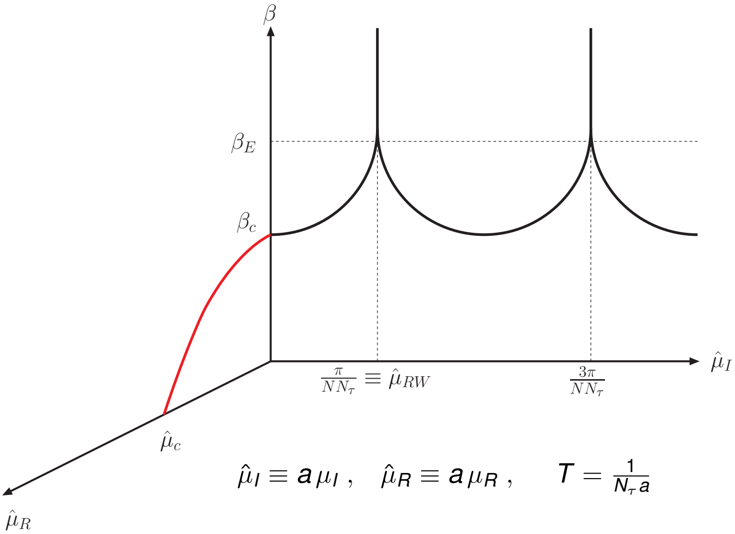

T
T_E
$\frac{\pi}{N}$
$\frac{3\pi}{N}$
$\frac{5\pi}{N}$
$\frac{7\pi}{N}$
θ


T
T_E
T_c
$\frac{\pi}{N}$
$\frac{3\pi}{N}$
$\frac{5\pi}{N}$
$\frac{7\pi}{N}$
θ

Analytic continuation of the critical line



$$\hat{\mu}_I \equiv a\,\mu_I\,, \quad \hat{\mu}_R \equiv a\,\mu_R\,, \qquad T = \frac{1}{N_\tau a}$$

The most important application of the method is the analytic continuation of the critical line itself.

Strategy

- locate the (pseudo-)critical β's for several fixed values of the imaginary chemical potential, by looking for peaks in the susceptibilities of a given observable
- interpolate the critical β's obtained at imaginary chemical potential with an analytic function of μ^2, to be then extrapolated to real chemical potential
- if the theory is free of the sign problem, compare the extrapolated curve with the determinations of the critical β's at real chemical potential.

Observables: chiral condensate, Polyakov loop, plaquette.

Investigations in QCD-like theories

- Early approaches in QCD: pseudocritical line $\beta_c(\mu^2)$ well interpolated by $\beta_c(\mu^2) = \beta_c(0) + A\mu^2$, for $\mu = i\mu_I$, at small μ_I

  [Ph. de Forcrand, O. Philipsen, 2002-2003]

  [M. D'Elia, M.P. Lombardo, 2003-2004]

- Later on, systematic investigations aimed at extending the domain of reliability of the method
  - wider range of μ_I values in numerical simulations
  - larger statistics
  - several trial interpolations

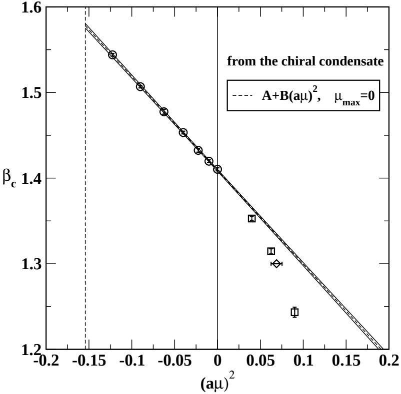
  [P. Cea, L. Cosmai, M. D'Elia, A.P. *et al*, 2006→]

Testfield: QCD-like theories (two-color QCD and finite isospin QCD) free of the sign problem, where the analytic continuation can be compared with Monte Carlo determinations obtained directly at real chemical potentials.

Outline

SU(2), $n_f = 8$ staggered, $16^3 \times 4$ lattice, $am = 0.07$

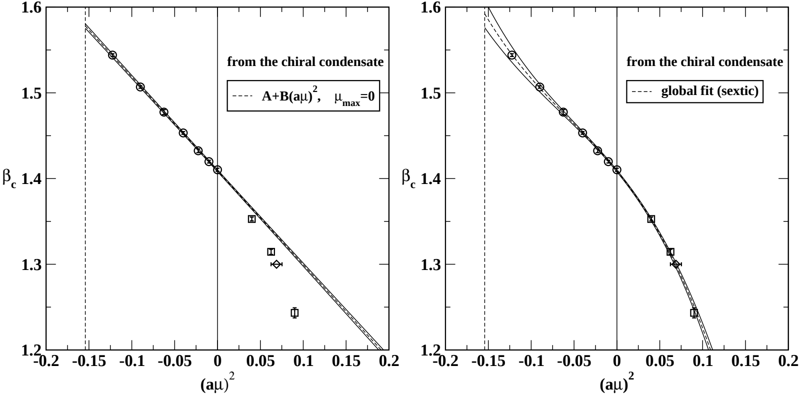
[P. Cea, L. Cosmai, M. D'Elia, A.P., Phys. Rev. D77 (2008) 051501]



No room for fitting functions different from $A + B\hat{\mu}^2$ at $\mu^2 < 0$;
extrapolation fails!

SU(2), $n_f = 8$ staggered, $16^3 \times 4$ lattice, $am = 0.07$

[P. Cea, L. Cosmai, M. D'Elia, A.P., Phys. Rev. D77 (2008) 051501]

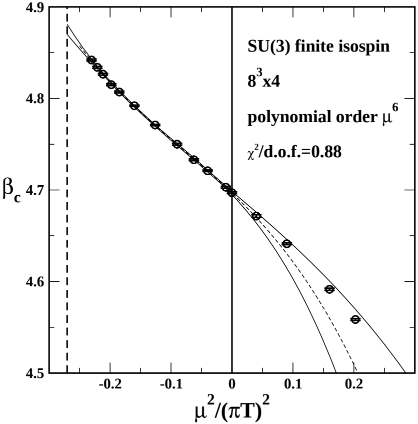


A global fit with $A_0 + A_1(a\mu)^2 + A_2(a\mu)^4 + A_3(a\mu)^6$ works nicely; remark: all $A_i > 0$.

Outline

Finite isospin SU(3), $n_f = 8$ staggered, $8^3 \times 4$ lattice, am=0.1

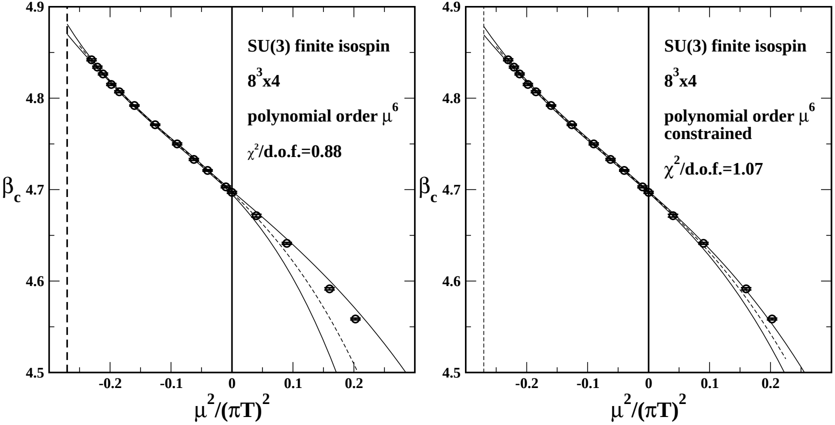
[P. Cea, L. Cosmai, M. D'Elia, C. Manneschi, A.P., Phys. Rev. D80 (2009) 034501]



Deviations from the linear behavior in μ^2 are evident at $\mu^2 < 0$. At least a 3rd order polynomial in μ^2 is needed; extrapolation OK.

Finite isospin SU(3), $n_f = 8$ staggered, $8^3 \times 4$ lattice, *am*=0.1

[P. Cea, L. Cosmai, M. D'Elia, C. Manneschi, A.P., Phys. Rev. D80 (2009) 034501]



Predictivity is increased if the coefficient of μ^2 in the 3rd order polynomial in μ^2 is constrained by a linear fit in the region near $\mu = 0$.

Outline

SU(3) with $n_f = 4$

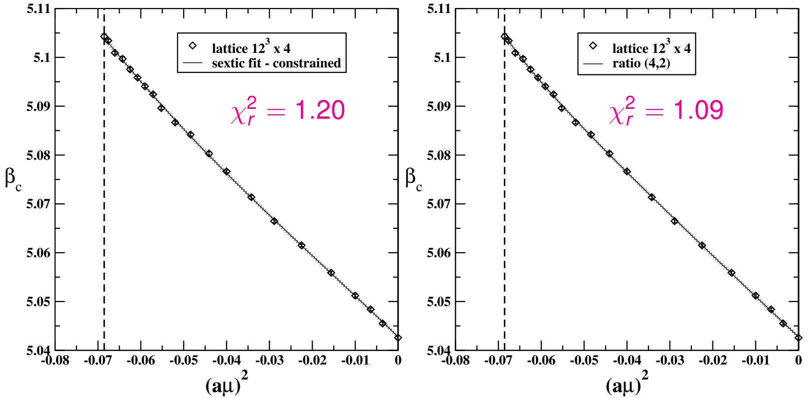
SU(3), $n_f = 4$ staggered, $12^3 \times 4$ lattice, $am = 0.05$
[P. Cea, L. Cosmai, M. D'Elia, A.P., Phys. Rev. D81 (2010) 094502]

Setup:

- Φ-hybrid Monte Carlo algorithm, with *dt*=0.01
  [S.A. Gottlieb *et al.*, 1987]
- statistics: 10k trajectories of 1 Molecular Dynamics unit
  (up to 100k for a few β's near $\beta_c(\mu^2)$)
- $\beta_c(\mu^2)$ determined as the position of the peak in the susceptibility of the (real part of) the Polyakov loop
- simulations on apeNEXT and on the PC cluster of the INFN Bari Computer Center for Science

SU(3), $n_f = 4$ staggered, $12^3 \times 4$ lattice, $am = 0.05$

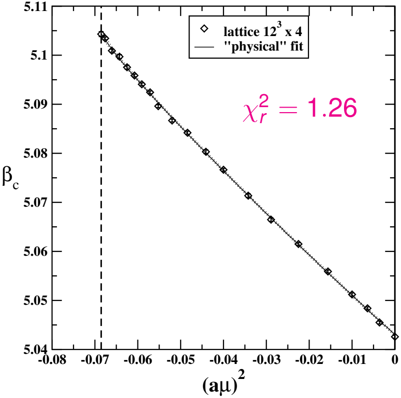
[P. Cea, L. Cosmai, M. D'Elia, A.P., Phys. Rev. D81 (2010) 094502]



- Deviations from the linear behavior in μ^2 are seen
- Also a plain 3rd order polynomial in μ^2 works well
- It is hard to see differences among the successful interpolations

SU(3), $n_f = 4$ staggered, $12^3 \times 4$ lattice, $am = 0.05$

[P. Cea, L. Cosmai, M. D'Elia, A.P., Phys. Rev. D81 (2010) 094502]



"Physical fit":

$$\left[\frac{T_c(\mu)}{T_c(0)}\right]^2 = \frac{1 + C\mu^2/T_c^2(\mu)}{1 + A\mu^2/T_c^2(\mu) + B\mu^4/T_c^4(\mu)}$$

$$a(\beta_c(\mu^2))^2 \,|_{2-\mathrm{loop}} = a(\beta_c(0))^2 \,|_{2-\mathrm{loop}} \times \frac{1 + A\mu^2/T_c^2 + B\mu^4/T_c^4}{1 + C\mu^2/T_c^2}$$

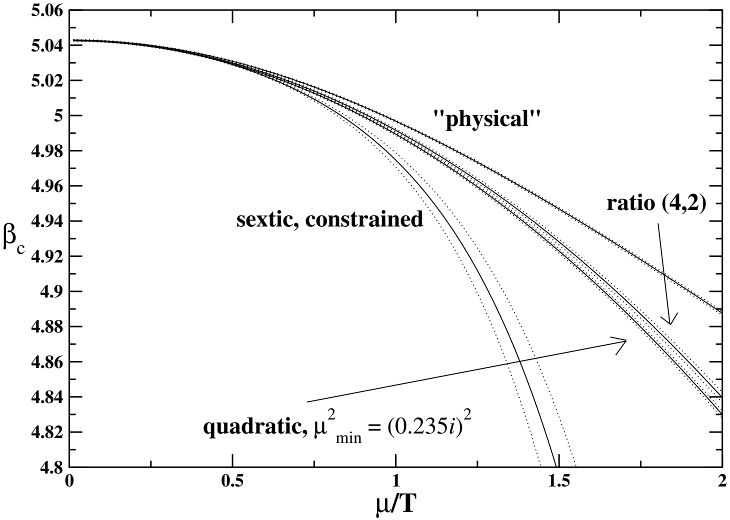
The formal limit $T_c \to 0$ leads to

$$\mu_c(T=0) = \sqrt{\frac{C}{B}}\,T_c(0) = 2.5904(93)\,T_c(0)$$

SU(3), $n_f = 4$ staggered, $12^3 \times 4$ lattice, $am = 0.05$

[P. Cea, L. Cosmai, M. D'Elia, A.P., Phys. Rev. D81 (2010) 094502]

Extrapolations to positive μ^2



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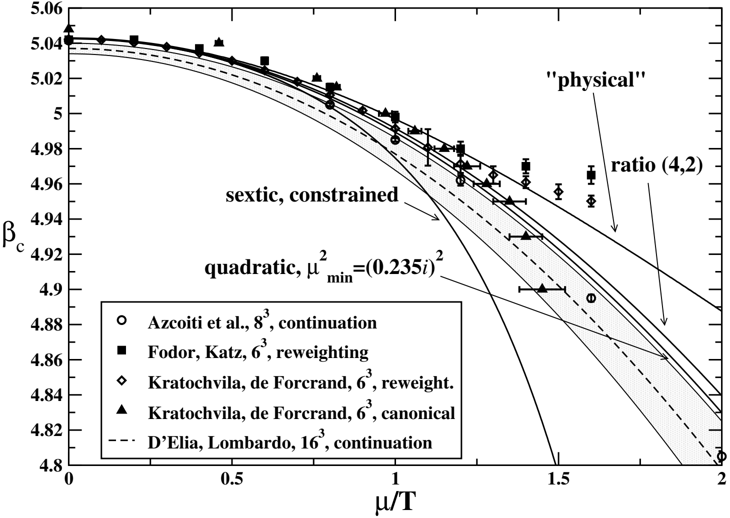

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SU(3) with $n_f = 2$

SU(3), $n_f = 2$ staggered, $16^3 \times 4$ lattice, $am = 0.05$
[P. Cea, L. Cosmai, M. D'Elia, A.P., F. Sanfilippo, in preparation]

Setup:

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  [S.A. Gottlieb *et al.*, 1987]
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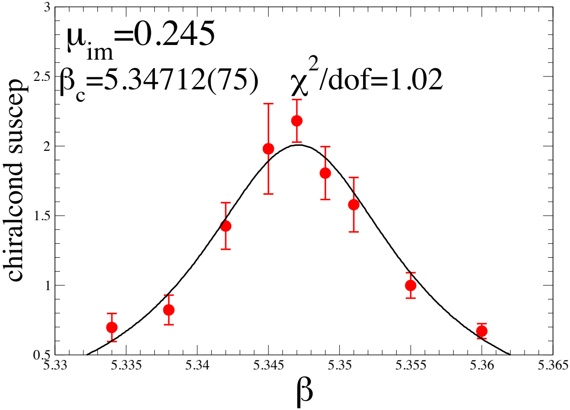
- $\beta_c(\mu^2)$ determined as the position of the peak in the susceptibility of the chiral condensate
- simulations on the PC clusters of the INFN Bari Computer Center for Science, INFN-Genova and INFN-Pisa

SU(3), $n_f = 2$ staggered, $16^3 \times 4$ lattice, $am = 0.05$ - PRELIMINARY
[P. Cea, L. Cosmai, M. D'Elia, A.P., F. Sanfilippo, in preparation]

An example of determination of the pseudocritical β.



SU(3), $n_f = 2$ staggered, $16^3 \times 4$ lattice, $am = 0.05$ - PRELIMINARY

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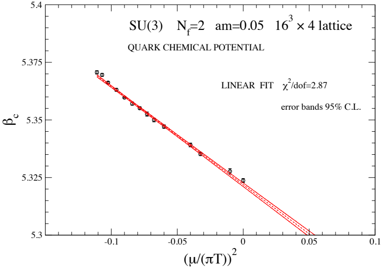
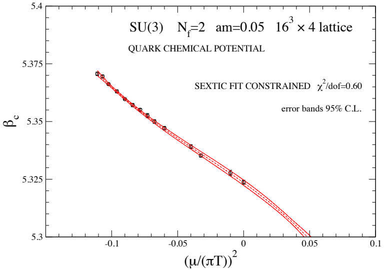
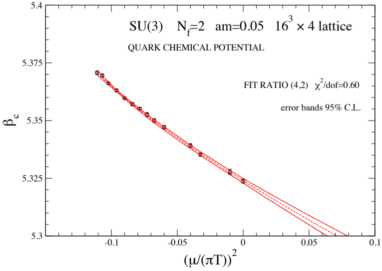






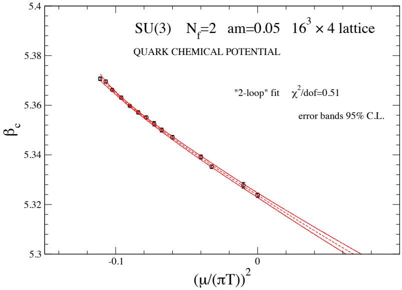
"Curvature" at $\mu = 0$:

$$\frac{d\beta_c(\mu^2)}{d\mu^2}\Big|_{\mu=0} = -0.52228(1021).$$

[Ph. de Forcrand, O. Philipsen, 2002]: $-0.50009(3375)$ for $am = 0.025$.

SU(3), $n_f = 2$ staggered, $16^3 \times 4$ lattice, $am = 0.05$ - PRELIMINARY

[P. Cea, L. Cosmai, M. D'Elia, A.P., F. Sanfilippo, in preparation]



"Physical fit":

$$\left[\frac{T_c(\mu)}{T_c(0)}\right]^2 = \frac{1 + C\mu^2/T_c^2(\mu)}{1 + A\mu^2/T_c^2(\mu) + B\mu^4/T_c^4(\mu)}$$

$$a(\beta_c(\mu^2))^2 \,|_{2-\text{loop}} = a(\beta_c(0))^2 \,|_{2-\text{loop}}$$

$$\times \frac{1 + A\mu^2/T_c^2 + B\mu^4/T_c^4}{1 + C\mu^2/T_c^2}$$

The formal limit $T_c \to 0$ leads to

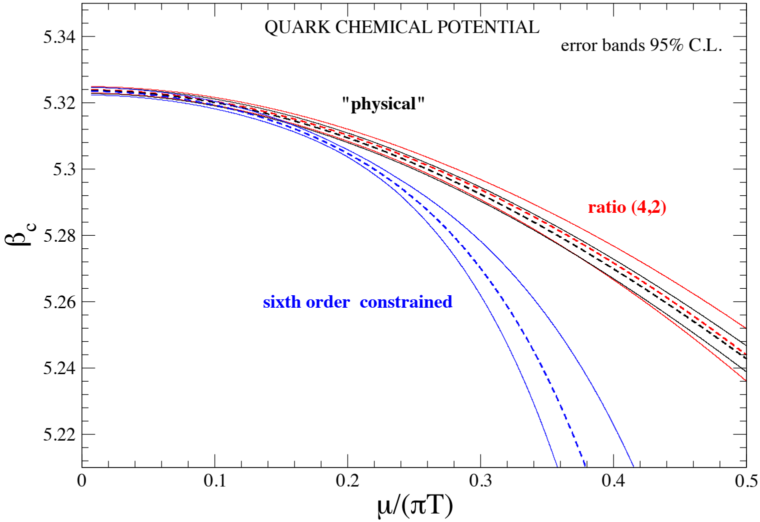
$$\mu_c(T=0) = \sqrt{\frac{C}{B}}\, T_c(0) = 3.284(64)\, T_c(0)$$

[A. Nagata, K. Nakamura, 2011]: $2.73(58)\, T_c(0)$ for $n_f = 2$ Wilson.

SU(3), $n_f = 2$ staggered, $16^3 \times 4$ lattice, $am = 0.05$ - PRELIMINARY

[P. Cea, L. Cosmai, M. D'Elia, A.P., F. Sanfilippo, in preparation]

Extrapolations to positive μ^2



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Finite isospin SU(3), $n_f = 2$ staggered, $16^3 \times 4$ lattice, $am = 0.05$ - PRELIMINARY

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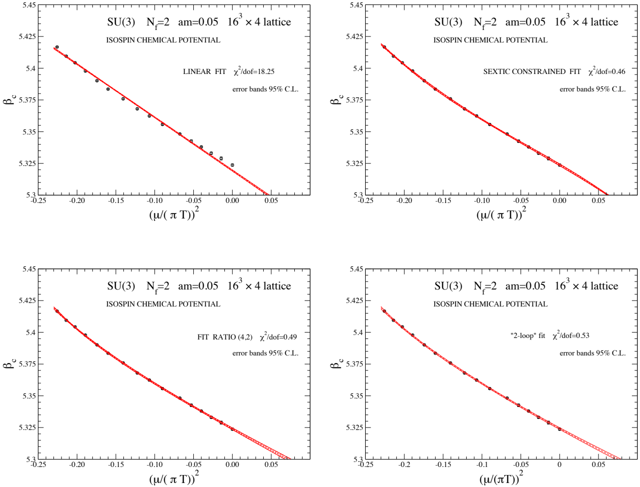

"Curvature" at $\mu = 0$: $\frac{d\beta_c(\mu^2)}{d\mu^2}\Big|_{\mu=0} = -0.47018(1270)$.

It is about 5σ away from the case of finite baryon density!

Finite isospin SU(3), $n_f = 2$ staggered, $16^3 \times 4$ lattice, $am = 0.05$ - PRELIMINARY

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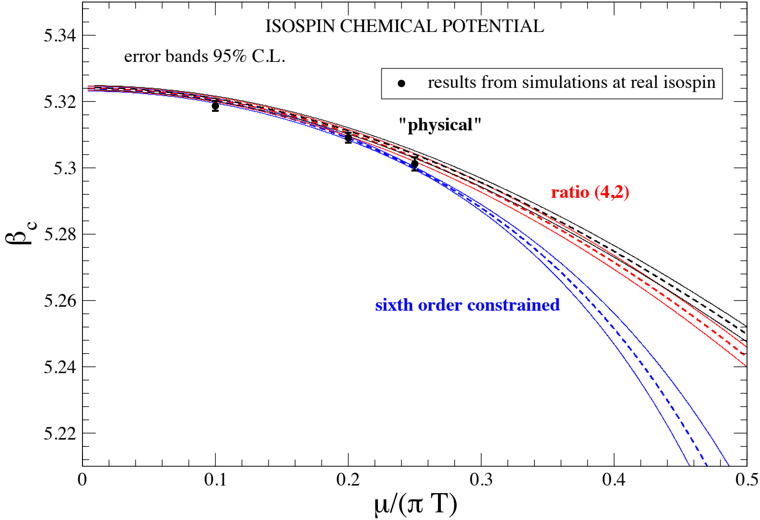
Extrapolations to positive μ^2



Conclusions

- Deviations from the quadratic behavior in μ of the pseudocritical couplings at negative μ^2 are clearly visible in QCD with $n_f = 2$ and 4.
- There are, however, several kinds of functions able to interpolate them, leading to extrapolations which diverge from each other at large real μ.
- The situation is quite similar in $n_f = 2$ QCD with non-zero isospin density. The curvature of the critical line at $\mu = 0$ is less pronounced here, than in $n_f = 2$ QCD with finite baryon density.
- The use of finer lattices and/or improved lattice actions could reduce the systematic effects involved in the method.

Jean Claude Richard de Saint-Non, "Vuë de la ville et du port de Bari", 1783
(Biblioteca Nazionale Marciana di Venezia)