Lattice calculations of isospin breaking corrections due to $m_u \neq m_d$

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why isospin breaking?

we do have a lot of precise experimental measurements in the quark flavour sector of the standard model that, combined with CKM unitarity (first row), allow us to **measure** hadronic matrix elements

a simple example from FLAVIAnet kaon working group **M.Antonelli et al. Eur.Phys.J.C69**

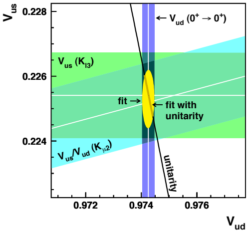
$$\begin{cases} \left|\frac{V_{us} F_K}{V_{ud} F_\pi}\right| = 0.27599(59) \\ \\ \left|V_{us} f_+^{K\pi}(0)\right| = 0.21661(47) \end{cases} \qquad \begin{cases} |V_{ud}|^2 + |V_{us}|^2 = 1 \\ \\ |V_{ud}| = 0.97425(22) \end{cases}$$

where $|V_{ud}|$ comes by combining 20 super-allowed nuclear β-decays and $|V_{ub}|$ has been neglected because smaller than the uncertainty on the other terms, combine to give

$$|V_{us}| = 0.22544(95)$$

$$f_+^{K\pi}(0) = 0.9608(46)$$

$$\frac{F_K}{F_\pi} = 1.1927(59)$$

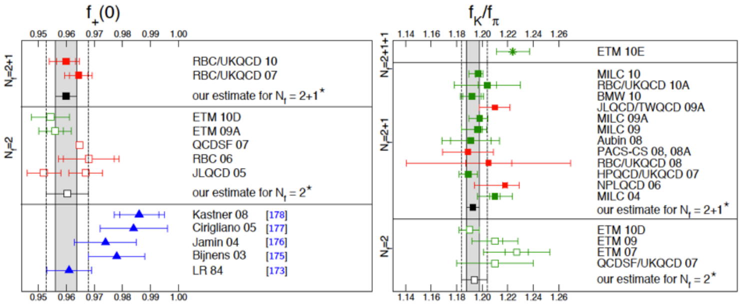


lattice QCD is **still** needed to **postdict** these quantities and, in case, to falsify the standard model

F_K/F_π & $f_+^{K\pi}(0)$ summary from FLAG

concerning theoretical predictions, and lattice QCD in particular, these matrix elements are among the well known quantities

G.Colangelo et al. arXiv:1011.4408



$$f_+^{K\pi}(0) = 0.956(8) \qquad \sim 0.8\% \qquad\qquad \frac{F_K}{F_\pi} = 1.193(5) \qquad \sim 0.5\%$$

to do better we should include effects that we have been neglecting up to now...

F_K/F_π & $f_+^{K\pi}(q^2)$ beyond the isospin limit

- there are two sources of isospin breaking effects,

$$\underbrace{m_u \neq m_d}_{\textbf{QCD}} \qquad\qquad \underbrace{e_u \neq e_d}_{\textbf{QED}}$$

- in the particular and (lucky) case of these observables, the correction to the isospin symmetric limit due to the difference of the up and down quark masses (**QCD**) can be estimated in *chiral perturbation theory*,

$$\begin{cases} f_+^{K\pi}(0) = 0.956(8) \qquad \sim 0.8\% \\ \left(\dfrac{f_+^{K^+\pi^0}(q^2)}{f_+^{K^0\pi^-}(q^2)} - 1 \right)_{QCD} = 0.029(4) \end{cases}$$

A. Kastner, H. Neufeld Eur.Phys.J.C57 (2008)

$$\begin{cases} \dfrac{F_K}{F_\pi} = 1.193(5) \qquad \sim 0.5\% \\ \left(\dfrac{F_{K^+}/F_{\pi^+}}{F_K/F_\pi} - 1 \right)_{QCD} = -0.0022(6) \end{cases}$$

V. Cirigliano, H. Neufeld arXiv:1102.0563

- we need first principle lattice QCD calculations to avoid uncertainties coming from the effective theory
- but the home message is: reducing the error on these quantities without taking into account isospin breaking is useless. . .

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the gauge configurations

β	am_{ud}^L	am_s^L	L/a	N_{conf}	a (fm)	$Z_P(\overline{MS}, 2GeV)$
3.80	0.0080	0.0194	24	150	0.0977(31)	0.411(12)
	0.0110		24	150		
3.90	0.0030	0.0177	32	150	0.0847(23)	0.437(07)
	0.0040		32	150		
	0.0040		24	150		
	0.0064		24	150		
	0.0085		24	150		
	0.0100		24	150		
4.05	0.0030	0.0154	32	150	0.0671(16)	0.477(06)
	0.0060		32	150		
	0.0080		32	150		
4.20	0.0020	0.0129	48	100	0.0536(12)	0.501(20)
	0.0065		32	150		

- gauge configurations for this study have been taken from the gauge ensambles made publicly available by the ETMC collaboration
- **caveat:** the Twisted Mass discretization breaks isospin at finite lattice spacing
- we have been working in a mixed-action setup by introducing $O(a^2)$ errors coming from violations of unitarity
- in what follows I shall illustrate our method without discussing these technical details by thinking to a isospin-symmetric lattice regularization

isospin breaking on the lattice

- the calculation of QED isospin breaking effects on the lattice it has been don for the first time in
  Duncan, Eichten, Thacker, Phys. Rev. Lett. 76 (1996)
- QED is treated in the quenched approximation in its "compact" formulation
- because the photons are massless and unconfined this approach may introduce large finite volume effects. . .
- we shall come back on QED effects later in this talk

- the calculation of QCD isospin breaking effects on the lattice poses a theoretical problem

$$
\begin{aligned}
Z &= \int DUD\psi\ e^{-S_g[U]+S_f[U;m_u,m_d]} \\
&= \int DU\ e^{-S_g[U]} \underbrace{\det(D[U]+m_u)\ \det(D[U]+m_d)}_{\text{must be } >0}
\end{aligned}
$$

- if $m_u \neq m_d$ this can be only achieved by recurring to non (ultra) local and, consequently, very expensive fermion formulations (overlap)
- furthermore the effect is very small and it can be extremely difficult to see it with limited statistical accuracy

our QCD isospin breaking on the lattice

- our idea is to calculate QCD isospin corrections **at first order** in $\varepsilon_{ud} = (m_d - m_u)/2$:

$$
\begin{aligned}
S &= \bar{u}\,(D[U] + m_u)\,u + \bar{d}\,(D[U] + m_d)\,d \\
&= \underbrace{\bar{u}\,(D[U] + m_{ud})\,u + \bar{d}\,(D[U] + m_{ud})\,d}_{S_0} - \overbrace{\frac{m_d - m_u}{2}(\bar{u}u - \bar{d}d)}^{\varepsilon_{\mathbf{ud}}\hat{\mathbf{S}}}
\end{aligned}
$$

- the calculation of an observable proceeds as follows

$$
\begin{aligned}
\langle\mathcal{O}\rangle - \boldsymbol{\Delta}\langle\mathcal{O}\rangle &= \frac{\int DU\ e^{-S_g[U]-S_0[U]+\varepsilon_{ud}\hat{S}}\ \mathcal{O}}{\int DU\ e^{-S_g[U]-S_0[U]+\varepsilon_{ud}\hat{S}}} = \frac{\int DU\ e^{-S_g[U]-S_f^0[U]}\ (1+\varepsilon_{ud}\hat{S})\ \mathcal{O}}{\int DU\ e^{-S_g[U]-S_f^0[U]}\ (1+\varepsilon_{ud}\hat{S})} \\
&= \langle\mathcal{O}\rangle + \varepsilon_{\mathbf{ud}}\langle\hat{\mathbf{S}}\ \mathcal{O}\rangle - \underbrace{\varepsilon_{ud}\langle\hat{S}\rangle}_{=0}
\end{aligned}
$$

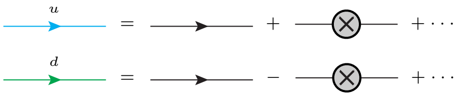
our QCD isospin breaking on the lattice

- to insert $\bar{u}u - \bar{d}d$ within a correlation function amounts (after Wick contractions) to calculate **the same observables** but with **light propagators squared**

$$\mathcal{S}_u = \frac{1}{D[U]+m_{ud}-\varepsilon_{ud}} = \frac{1}{D[U]+m_{ud}} + \frac{\varepsilon_{ud}}{(D[U]+m_{ud})^2}$$

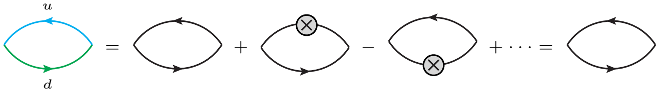
$$\mathcal{S}_D = \frac{1}{D[U]+m_{ud}+\varepsilon_{ud}} = \frac{1}{D[u]+m_{ud}} - \frac{\varepsilon_{ud}}{(D[U]+m_{ud})^2}$$

- relations that can be represented diagrammatically as



our QCD isospin breaking on the lattice: two point functions

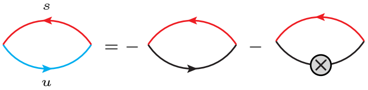
- at first order in ε_{ud} pion mass and decay constants don't get a correction (here $\pi^{\pm}$ but it works also for π^0 because $\langle \pi \| \hat{S} \| \pi \rangle = \langle 1, I_3 \| 1, 0 \| 1, I_3 \rangle = 0$)



$+\ \mathcal{O}(\varepsilon_{ud}^2)$

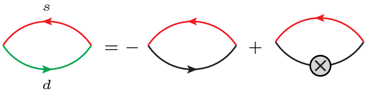
- the kaons do get a correction

$C_{K^+K^-}(t) \quad = \quad -$



$+\ \mathcal{O}(\varepsilon_{ud}^2)$

$C_{K^0K^0}(t) \quad = \quad -$



$+\ \mathcal{O}(\varepsilon_{ud}^2)$

- this means that at first order ($\delta_{.}$ stays for relative error while $\Delta_{.}$ for absolute error),

$$\delta_u \left(\frac{F_K}{F_\pi} \right) = \frac{\Delta_u F_K}{F_K} - \frac{\Delta_u F_\pi}{F_\pi} = \frac{F_K - F_{K^+}}{F_K}$$

what do we expect from "corrected" correlation functions?

let's consider the euclidean correlation function in the **full perturbed** theory, $C_{K^+K^-}(t)$, and in the **symmetric unperturbed** theory, $C_{KK}(t)$:

$$
\begin{aligned}
C_{K^+K^-}(t) &= \sum_{\vec{x}} \langle \bar{u}\gamma_5 s(\vec{x},t)\, \bar{s}\gamma_5 u(0)\rangle = \sum_n \langle 0|\bar{u}\gamma_5 s(0)|n^{\varepsilon_{ud}}\rangle \langle n^{\varepsilon_{ud}}|\bar{s}\gamma_5 u(0)|0\rangle\, e^{-E_n^{\varepsilon_{ud}} t} \\
&= \frac{G_{K^+}^2}{2E_{K^+}}\, e^{-E_{K^+}t} + \cdots
\end{aligned}
$$

$$
C_{KK}(t) = \frac{G_K^2}{2E_K}\, e^{-E_K t} + \cdots
$$

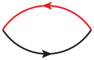
where the fact that the leading exponential is the same **is not obvious** and follows from the fact that our perturbation $\hat{S}$ is **flavour diagonal** (e.g. does not happen for insertions of the weak hamiltonian)

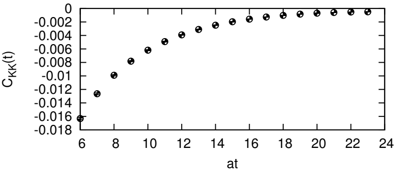
by using non degenerate perturbation theory (I_3 is conserved), we have

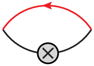
$$
E_{K^+} = E_K - \Delta E_K = E_K + \varepsilon_{ud}\langle K|\hat{S}|K\rangle
$$

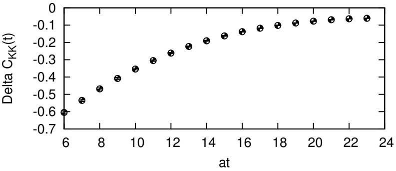
$$
|K^+\rangle = |K\rangle - |\Delta K\rangle = |K\rangle + \varepsilon_{ud}\sum_{n\neq K}|n\rangle \frac{\langle n|\hat{S}|K\rangle}{E_K - E_n}
$$

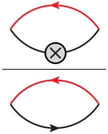
what do we expect from “corrected” correlation functions?

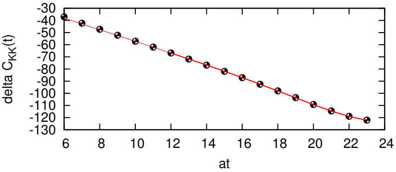
$-$  $= \dfrac{G_K^2}{2E_K}\; e^{-E_K t}$



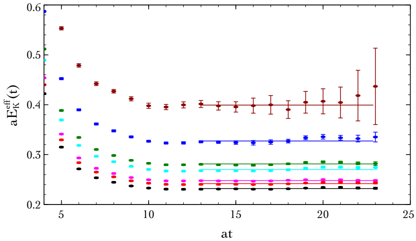
 $= \dfrac{G_K^2}{2E_K}\; e^{-E_K t}\left[\dfrac{\Delta(G_K^2/2E_K)}{G_K^2/2E_k} - t\Delta E_K\right]$



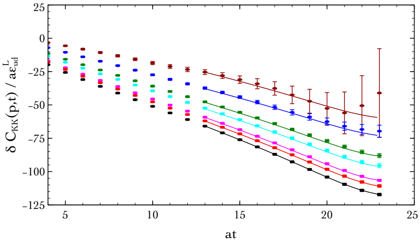
$-$  $= \delta\left(\mathbf{\dfrac{G_K^2}{2E_K}}\right) - \mathbf{\Delta E_K t}$



our QCD isospin breaking on the lattice: kaons two point functions



$$E_K^2(p) = M_K^2 + p^2$$



$$\Delta E_K(p) = \frac{M_K \Delta M_K}{\sqrt{M_K^2 + p^2}}$$

- by considering pseudoscalar-pseudoscalar correlators and by taking into account the finite time extent of the lattice, we fit correlations at different $\vec{p}$ according to,

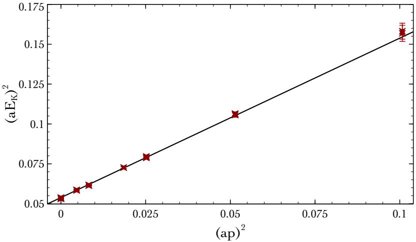
$$\delta C_{KK}(\vec{p}, t) = \delta \left(\frac{G_K^2 e^{-E_K T/2}}{2E_K} \right) + \Delta E_K (t - T/2) \tanh\left[E_K (t - T/2)\right] + \ldots$$

- and extract F_K and δF_K according to

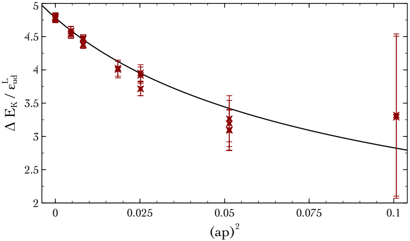
$$F_K = (m_s + m_{ud}) \frac{G_K}{M_K^2} \qquad \delta F_K = \frac{\varepsilon_{ud}}{m_s + m_{ud}} + \delta G_K - 2\delta M_K$$

our QCD isospin breaking on the lattice: kaons two point functions

are we sure that the slopes correspond to ΔE_K?



$$E_K^2(p) = M_K^2 + p^2$$



$$\Delta E_K(p) = \frac{M_K \Delta M_K}{\sqrt{M_K^2 + p^2}}$$

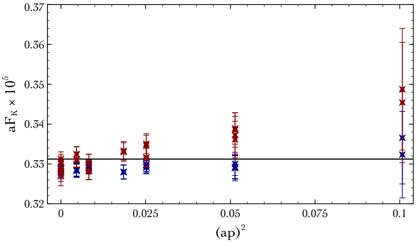
- the solid lines are not fitted, but theoretically predicted by using calculated M and ΔM
- this kind of accuracy on kinematics at $p \neq 0$ is possible thanks to the use of twisted boundary conditions

G.M. de Divitiis, R. Petronzio, N.T. Phys.Lett. B595 (2004)

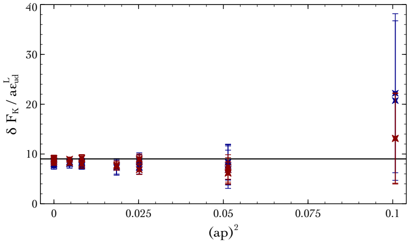
$$\psi(x+L) = e^{i\theta}\psi(x) \qquad \longrightarrow \qquad p = \frac{\theta}{L} + \frac{2\pi n}{L}$$

our QCD isospin breaking on the lattice: kaons two point functions

are we sure that the slopes intercepts to δF_K?



$$F_K(p) = F_K(M_K^2)$$



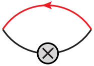
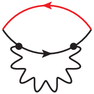
$$\delta F_K(p) = \delta F_K(M_K^2)$$

- the solid lines are not fitted, but drawn by using $F_K(p=0)$ and $\delta F_K(p=0)$
- this kind of accuracy on kinematics at $p \neq 0$ is possible thanks to the use of twisted boundary conditions
  G.M. de Divitiis, R. Petronzio, N.T. Phys.Lett. B595 (2004)

$$\psi(x+L) = e^{i\theta}\psi(x) \qquad \longrightarrow \qquad p = \frac{\theta}{L} + \frac{2\pi n}{L}$$

extracting $[m_d - m_u]^{QCD}$: QED corrections

- in order to extract $2\varepsilon_{ud}^{QCD} = [m_d - m_u]^{QCD}$ we need experimental inputs and we cannot neglect QED corrections
- If we work at first order in the QED coupling constant and ε_{ud} and neglect terms of $\mathcal{O}(\alpha_{em}\varepsilon_{ud})$, the relevant Feynman diagrams entering kaons two point functions are

$\Delta C_{KK}(t) =$  $- \frac{e_d^2 - e_u^2}{2}$  $- e_s \frac{e_d - e_u}{}$  $+ \mathcal{O}(\alpha_{em}\varepsilon_{ud})$

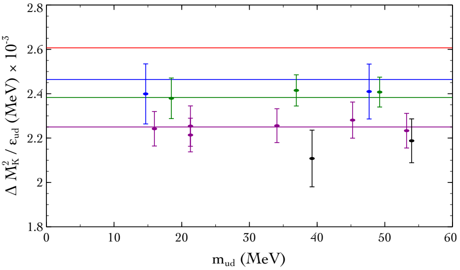
- the electromagnetic corrections to $C_{KK}(t)$ are logarithmically divergent, corresponding to the renormalization of the quark masses, and the separation of QED and QCD effects is ambiguous (**prescription dependent**)
- **in the chiral limit** QED corrections to $M^2_{K^0} - M^2_{K^+}$ and $M^2_{\pi^0} - M^2_{\pi^+}$ are the same (Dashen's theorem)
- **beyond the chiral limit** violations to Dashen's theorem are parametrized in term of small parameters

ε_γ **from FLAG: G.Colangelo et al. arXiv:1011.4408**

$\varepsilon_\gamma = 0.7(5) \quad \longleftarrow \quad$ our prescription

$$\left[M^2_{K^0} - M^2_{K^+}\right]^{QCD} = \left[M^2_{K^0} - M^2_{K^+}\right]^{exp} - (1+\varepsilon_\gamma)\left[M^2_{\pi^0} - M^2_{\pi^+}\right]^{exp} = 6.05(63) \times 10^3 \text{ MeV}^2$$

extracting $[m_d - m_u]^{QCD}$: chiral-continuum extrapolations



$$[m_d - m_u]^{QCD}\,(\overline{MS}, 2GeV) = 2\varepsilon_{ud}^{QCD}$$

$$= 2.29(5)(24)\ \text{MeV}$$

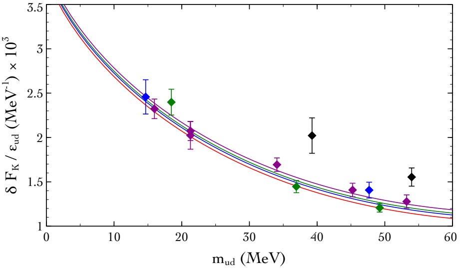
chiral perturbation theory formulae can be derived from known results

$n_f = 2 + 1$: **Gasser and Leutwyler Nucl. Phys. B250(1985)**
non unitary $n_f = 2$: **S.Sharpe Phys. Rev. D56(1997)**

$$\frac{\Delta M_K^2}{\varepsilon_{ud}} = B_0 \left\{ 1 + 2(m_{ud} + m_s)\hat{B}_0(2\alpha_8 - \alpha_5) + 4m_{ud}\hat{B}_0(2\alpha_6 - \alpha_4) \right.$$
$$\left. + \hat{B}_0 m_s \log(2\hat{B}_0 m_s) + \hat{B}_0 \frac{m_s + m_{ud}}{m_s - m_{ud}} \left[m_s \log(2\hat{B}_0 m_s) - m_{ud} \log(2\hat{B}_0 m_{ud}) \right] \right\}$$

where α_i are low energy constants and $\hat{B}_0 = 2B_0/(4\pi F_0^2)$

calculating δF_K^{QCD}: chiral-continuum extrapolations



$$\left[\frac{F_{K^+}/F_{\pi^+}}{F_K/F_\pi} - 1\right]^{QCD} = -0.00376(29)(4)$$

to be compared with

$$\left[\frac{F_{K^+}/F_{\pi^+}}{F_K/F_\pi} - 1\right]^{\chi pt} = -0.0022(6)$$

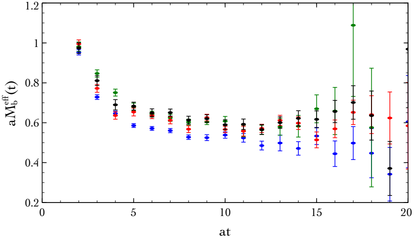
chiral perturbation theory formulae can be derived from known results

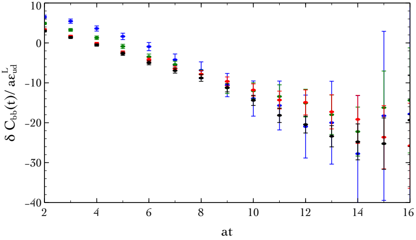
$n_f = 2+1$: Gasser and Leutwyler Nucl. Phys. B250(1985)
non unitary $n_f = 2$: S.Sharpe Phys. Rev. D56(1997)

$$\frac{\delta F_K}{\varepsilon_{ud}} = \frac{B_0}{2}\left\{\alpha_5 - \hat{B}_0 \frac{1}{m_s - m_{ud}}\left[m_s \log(2\hat{B}_0 m_s) - m_{ud}\log(2\hat{B}_0 m_{ud})\right]\right\}$$

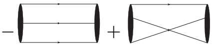
where α_i are low energy constants and $\hat{B}_0 = 2B_0/(4\pi F_0^2)$

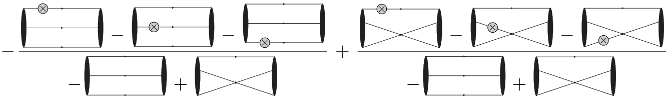
calculating $M_n - M_p$





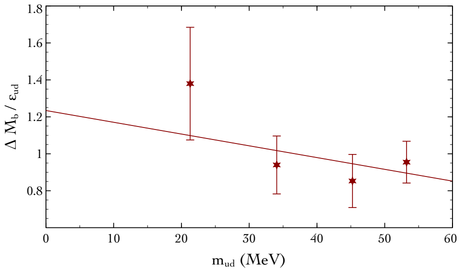
the calculation of the neutron-proton mass difference proceeds along the same lines as in the K^0-K^+ case

$C_{NN}(t) \quad = \quad$  $= W_b e^{-M_b t} + \cdots$

$\delta C_{bb}(t) \quad = \quad$ 

$\quad = \quad \delta W_b - t\Delta M_b + \cdots$

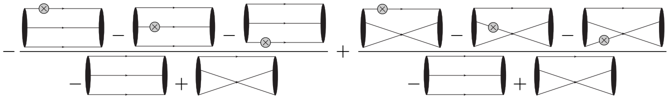
calculating $M_n - M_p$



$$\left[M_n - M_p\right]^{QCD} = 2\varepsilon_{ud}^{QCD}\left[\frac{\Delta M_b}{\varepsilon_{ud}}\right]^{QCD}$$

$$= 2.8(8)(3)\ \text{MeV}$$

$\delta C_{bb}(t) \quad =$



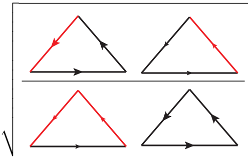
$$= \quad \delta W_b - t\Delta M_b + \cdots$$

- here the results are at fixed lattice spacing $a = 0.085$ fm.
- correlators have been compute by "Gaussian smearing" sink operators

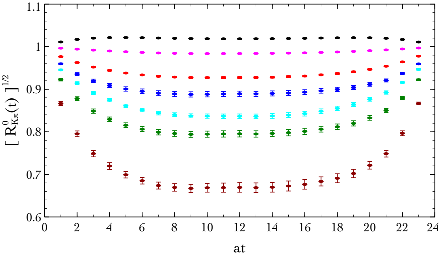
calculating $\delta f_+^{K\pi}(q^2)$

form factors parametrizing semileptonic decays can be calculated with good precision by considering **double ratios** of three point correlation functions

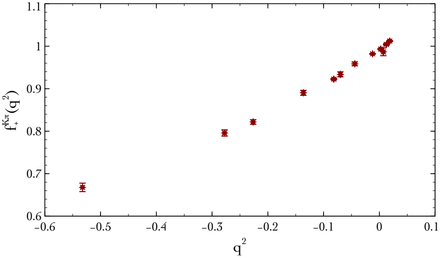
$$\frac{\langle\pi|V^\mu_{su}|K\rangle}{2\sqrt{E_\pi E_K}} =$$



$=$



and



$$\langle\pi|V^0_{su}|K\rangle = (E_K + E_\pi)f_+^{K\pi} + (E_K - E_\pi)f_-^{K\pi}$$

$$\langle\pi|\vec{V}_{su}|K\rangle = (\vec{p}_i + \vec{p}_f)f_+^{K\pi} + (\vec{p}_i - \vec{p}_f)f_-^{K\pi}$$

calculating $\delta f_+^{K\pi}(q^2)$

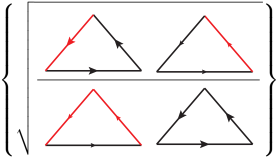
in order to calculate QCD isospin breaking corrections to $K \longrightarrow \pi\ell\nu$ form factors one needs to calculate,

$$\langle\pi|\mathrm{T}\left\{\int d^4x\ S^3(x;\mu)\ V^\mu_{su}\right\}|K\rangle \quad\longrightarrow\quad \begin{cases} \langle\bar{K}|\mathrm{T}\left\{\int d^4x\ H_W^{\Delta S=1}(x;\mu)\ H_W^{\Delta S=1}(0;\mu)\right\}|K\rangle \\ \\ \langle\pi|\mathrm{T}\left\{\int d^4x\ H_W^{\Delta S=1}(x;\mu)\ V^\mu_{em}\right\}|K\rangle \end{cases}$$

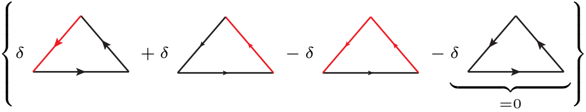
a key difference with respect to the calculation of long distance effects for $K \longrightarrow \pi\nu\nu$ and K-$\bar{K}$ mixing is that the isospin breaking correction does not induce the decay of the kaon. . .

by using perturbation theory it can be shown that the isospin breaking corrections to the matrix elements is given by (all **t-dependent** and wave function contributions **cancel**)

$$\delta\left\{\frac{\langle\pi|V^\mu_{su}|K\rangle}{2\sqrt{E_\pi E_K}}\right\} = \delta$$

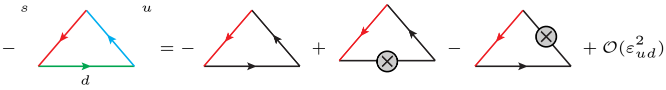


$$= \frac{1}{2}$$

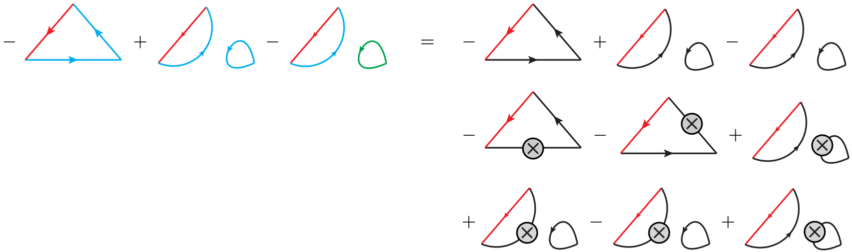


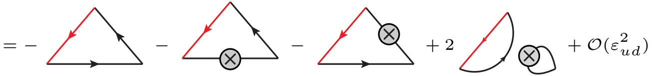
calculating $\delta f_+^{K\pi}(q^2)$

the diagrammatic expansion in the $K^0 \to \pi^- \ell\nu$ is

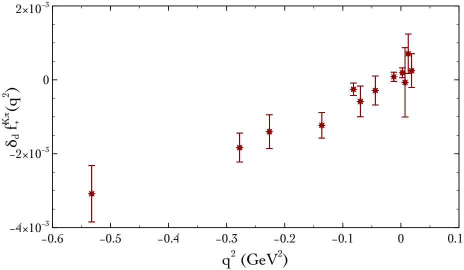


and is different, because of the **disconnected diagrams**, from the $K^+ \to \pi^0 \ell\nu$ case

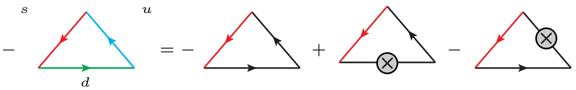




calculating $\delta f_+^{K\pi}(q^2)$



$$\left[\frac{f_+^{K^0\pi^-}(0) - f_+^{K\pi}(0)}{f_+^{K\pi}(0)}\right]^{QCD} = 1.9(4)(2) \times 10^{-4}$$



$$- \cdots = - \cdots + \cdots - \cdots + \mathcal{O}(\varepsilon_{ud}^2)$$

- in this work we have **not calculated disconnected diagrams**
- we can only show results for the $K^0 \to \pi^- \ell\nu$ case (above)
- this is a quantity that cannot be measured directly and the missing contribution, according to χpt, is expected to be much bigger
- the results given here make us confident on the possibility of completing the calculation by including disconnected diagrams

outlooks

- first results obtained by applying our method look very promising
- the method is general and can be applied to many observables, even at second order: we plan to apply it to $M_{\pi^+} - M_{\pi^0}$
- we shall also refine our results in the case of nucleon masses and form factors
- and compute QED effects by ourself
- first small steps toward the calculation of other observables that are relevant for phenomenological applications (long distance effects, etc.)