Levy walks in quenched disordered media

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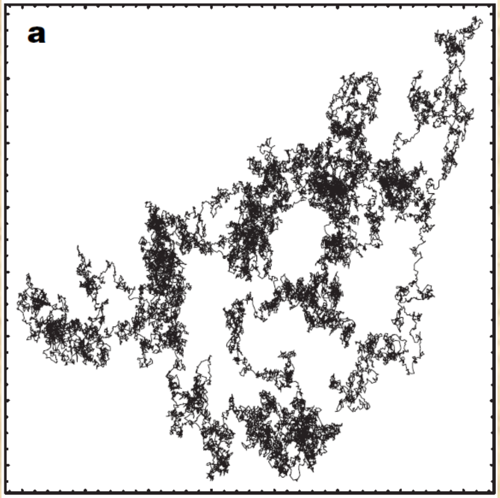
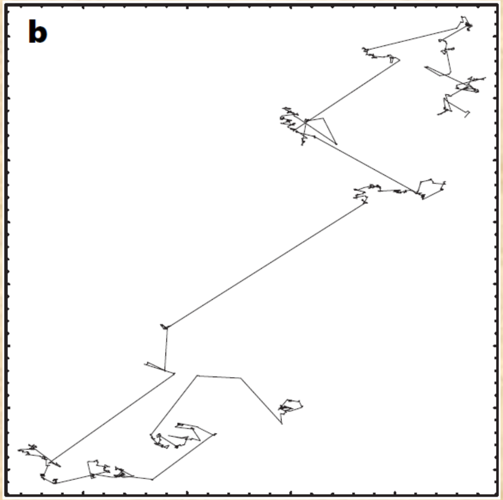
LEVY PROCESSES





Brownian Motion:
basic model for diffusion.
Sequence of steps of bounded length and random direction

Lévy walks:
Sequence of steps of unbounded length and random direction. The probability of a long jump of length *r* decays as a power low $r^{-\varepsilon}$

LEVY WALKS are SUPERDIFFUSIVE PHENOMENA
characterized by anomalous exponents:

Mean square displacement $\langle r^2(t) \rangle \approx t^{\gamma}$ **with** $\gamma > 1$

Resistence R, **Trasmittivity** T $R \approx 1/T \approx r^{\delta}$ **with** $\delta < 1$

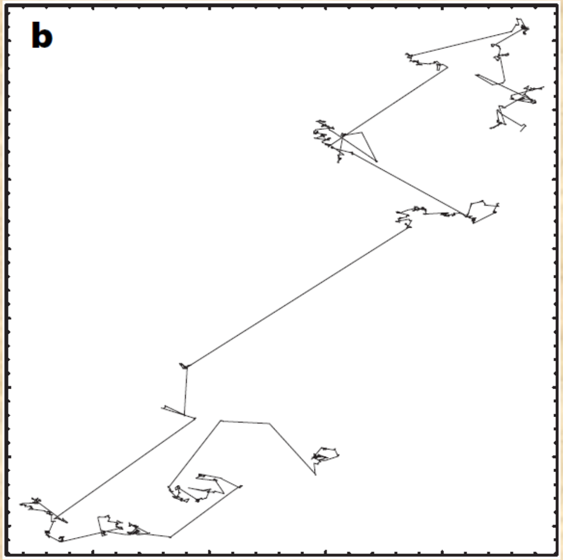
r **distance**

t **time**

LEVY WALKS AND FLIGHTS

Stocastic process: walker steps from $\vec{r}$ to $\vec{r}\,'$, transition probability

for large distances: $p(\vec{r}\,|\,\vec{r}\,') \approx |\vec{r}\,'-\vec{r}\,|^{-d-\alpha}$ $\alpha > 0$ **(long jumps are possible)**



Annealed Lévy Flights

Each step is covered in the same time independently of the distance $|\vec{r}\,'-\vec{r}\,|$

Annealed Lévy Walks

Each step is covered with constant velocity v, i.e. in a time proportional to the distance $|\vec{r}\,'-\vec{r}\,|$

Exact results for Annealed Lévy Walks

$$\langle r^2(t)\rangle \approx \begin{cases} t^2 \ \textbf{for} \ 0<\alpha<1 \\ t^{3-\alpha} \ \textbf{for} \ 1<\alpha<2 \\ t \ \textbf{for} \ \alpha>2 \end{cases}$$

$$R(r) \approx \begin{cases} r^{\alpha/2} \textbf{for} \ \alpha<2 \\ r \ \textbf{for} \ \alpha>2 \end{cases}$$

EXPERIMENTS

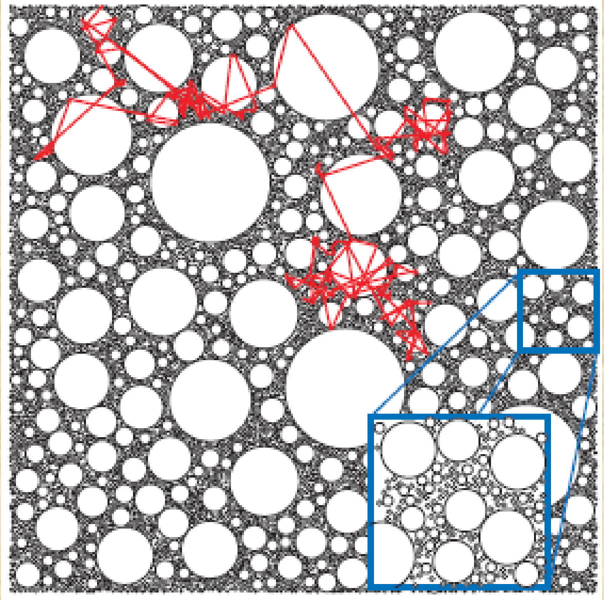
Lévy processes have been applied in different fields: biological systems, ecology, turbulent fluids, porous media, human travels and geology.

Experimental realization with controlled parameters

P. Barthelemy, J. Bertolotti, and D. S. Wiersma LENS Florence 2008

Glass spheres with diameters chosen according a Lévy distribution are packed into a matrix of scattering material the light ray performs long jumps across the spheres and is randomly deflected by the scaterers.

Typical Lèvy behaviors have been measured e.g. the transmission coefficient and its deviations.



In the experiment long tail are realized using spheres of diameters ranging from 500 μm to 5 μm.

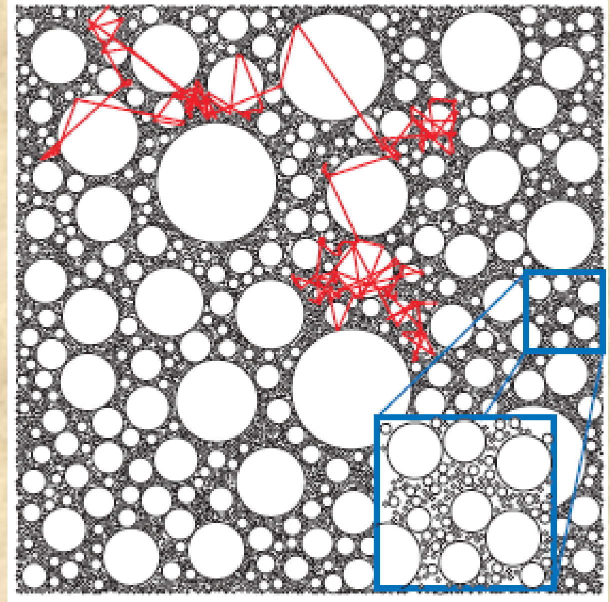
TOPOLOGICAL CORRELATIONS

LENS experiments have been interpreted by means of annealed Lévy walk: Light moves at constant velocity in the spheres, ok.

MAIN PROBLEM

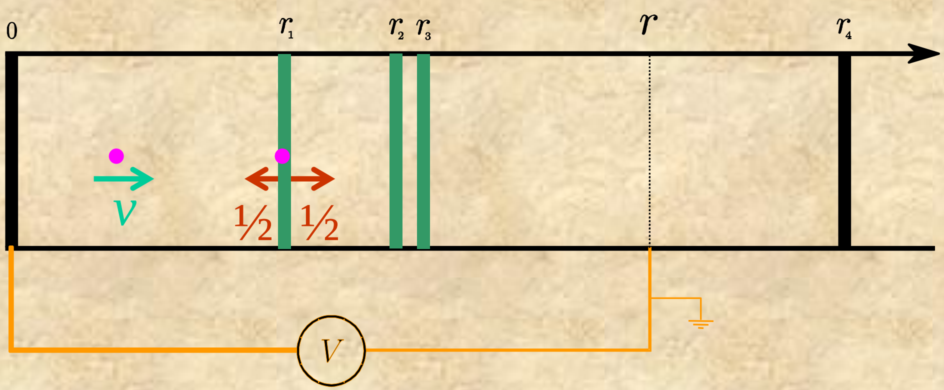
In Annealed Lévy-Walks the lengths of jumps are randomly chosen at each step i.e. they do not depend on previous moves. Therefore steps are uncorrelated.

In LENS experiments the step length is induced by sample topology and therefore correlated. E.g. after crossing a large sphere there is a high probability of being back scattered.



Inorder to explain the experiment a theory characterized by a quenched Lévy distribution of step length is required

1-DIMENSIONAL QUENCHED MODEL



Scatteres placed in $o, r_1, r_2, \ldots.$ **spaced according to a Lévy distribution e.g.**

$$p(r_{i+1} - r_i) = \frac{\alpha r_0^{\alpha}}{|r_{i+1} - r_i|^{1+\alpha}} \text{ if } |r_{i+1} - r_i| > r_0$$

$$p(r_{i+1} - r_i) = 0 \text{ if } |r_{i+1} - r_i| > r_0 \text{ otherwise,}$$

r_0 **cutoff fixing the space scale,**
α **characterizes Lévy distribution**

LEVY WALK: the particle moves ballistically (with constant velocity v) until it reaches a scatterer where it is transmitted or reflected with probability ½ [1] E. Barkai, V. Fleurov, J. Klafter, Phys. Rev. E 61 1164 (2000)

ELECTRIC MODEL: the resistance $R(r)$ between two contacts at distance r is the number of scatterers separating them [2] C.W.J. Beenakker, C.W. Groth, A.R. Akhmerov, Phys. Rev. B 79, 024204 (2009).

Known results

Long Levy-tails: Different average procedures provides Different results [1]

Average performed placing random-walk starting site or electric contacts

in any point of the structure

$$R(r) \approx \begin{cases} 0 \textbf{ for } \alpha < 1 \\ r \textbf{ for } \alpha > 1 \end{cases}$$

$$\langle r^2(t) \rangle \approx \begin{cases} t^2 \textbf{ for } \alpha < 1 \\ t^{3-\alpha} \textbf{ for } 1 < \alpha < 2 \\ t \textbf{ for } \alpha > 2 \end{cases}$$

Resistance [2]

Mean square displacement [1]

in a scattering point

$$R(r) \approx \begin{cases} r^{\alpha} \textbf{for } \alpha < 1 \\ r \textbf{ for } \alpha > 1 \end{cases}$$

$$\langle r^2(t) \rangle \approx \left\{ \; ? \right.$$

We complete the 1-dimensional picture evaluating $<\mathbf{r^2(t)}>$ averaged over scattering sites. Moreover we provide a general framework for the problem

LENS experiment light enters in the system with a scattering event
Averages performed considering scattering sites as starting point

SCALING RELATIONS

$P(r,t)$ **Average Probability for a walker to be in** r **after** t **steps,**

in 1dimension scaling hypothesis with $\ell(t)$ **characteristic lenght of** $P(r,t)$ **is**:

$$P(r,t) \approx \ell(t)^{-1} f(r/\ell(t)) + g(r,t)$$

Prefactor $\ell(t)^{-1}$ **provides 1d-normalization**

$\int \ell(t)^{-1} f(r/\ell(t)) dr = 1.$

$\lim_{t\to\infty} \int |P(r,t) - \ell(t)^{-1} f(r/\ell(t))| = 0,$

$g(r,t) \neq 0$ **only for** $r >> \ell(t)$ **i.e.**

$g(r,t)$ **represent a long tail of the distribution function**

The resistence can be evaluated as: $R(r) \approx \lim_{\omega\to 0} \tilde{P}(r,\omega).$

$\tilde{P}(r,\omega)$ **Fourier tranform of** $P(r,t) - P(0,t)$ **respect to** t

[3] M.E. Cates, J. Physique 46, 1059, (1985).

From scaling hypothesis follows the scaling relation:

$$\ell(t) \approx t^{1/z} \Leftrightarrow R(r) \approx r^{z-1}$$

In [2] Resistance of 1-dimensional calculated analyticaly static problem easier than dynamic random walk:

$$R(r) \approx \begin{cases} r^{\alpha} \textbf{ for } \alpha < 1 \\ r \textbf{ for } \alpha > 1 \end{cases}$$

Finaly:

$$\ell(t) \approx \begin{cases} t^{1/(1+\alpha)} \textbf{ for } \alpha < 1 \\ t^{1/2} \textbf{ for } \alpha > 1 \end{cases}$$

LONG TAILS AND ANOMALIES

Within scaling framework: Normal behavior $\left\langle r(t)^p \right\rangle \approx \ell(t)^p$

Single long-jump hypothesis:

Anomalies when dominates $g(r,t)$ i.e. the regime $r >> \ell(t)$. Tails of $P(r,t)$ provides a significant contributions to the mean square displacement:

$$P(r,t) \approx N(t) \cdot r^{-1-\alpha}$$

Probability for a scatterer of being followed by a jump at distance $r >> \ell(t)$

Number of scatterers visited by the walker in a time t **(definition of** R**)**

$$N(t) \approx \begin{cases} \ell(t)^\alpha \text{ if } \alpha < 1 \\ \ell(t) \text{ if } \alpha > 1 \end{cases}$$

i.e.:

$$N(t) \approx \begin{cases} t^{\frac{\alpha}{1+\alpha}} \text{ if } \alpha < 1 \\ t^{\frac{1}{2}} \text{ if } \alpha > 1 \end{cases}$$

For $\alpha < 1$ **and** $r >> \ell(t)$

$$P(r,t) \approx t^{\alpha/(1+\alpha)} r^{-1-\alpha} = \frac{1}{\ell(t)} \cdot \left(\frac{r}{\ell(t)} \right)^{-1-\alpha} \quad \text{a)}$$

For $\alpha > 1$ **and** $r >> \ell(t)$

$$P(r,t) \approx t^{\alpha/2} r^{-1-\alpha} = \frac{t^{\frac{1-\alpha}{2}}}{\ell(t)} \cdot \left(\frac{r}{\ell(t)} \right)^{-1-\alpha} \quad \text{b)}$$

MEAN SQUARE DISPLACEMENT

Moments of the average displacement

$$\left\langle r^p(t)\right\rangle = \int P(r,t) r^2 dr \approx \ell(t)^p + \int_{\ell(t)}^{vt} r^p N(t)\cdot r^{-1-\alpha} dr$$

Contributions to the first integral of distances $r < \ell(t)$

Contributions to the first integral of distances $r > \ell(t)$, vt is the natural upper cut off since particles can reach in a time t at most distance vt

Comparing the two terms we get:

$$\left\langle r^p(t)\right\rangle \approx \begin{cases} t^{\frac{p}{1+\alpha}} = \ell^p(t) \text{ for } \alpha < 1 \quad p < \alpha \\ t^{p-\frac{\alpha^2}{1+\alpha}} \text{ for } \alpha < 1 \quad p > \alpha \\ t^{p-\alpha+\frac{1}{2}} \text{ for } \alpha > 1 \quad p > 2\alpha - 1 \\ t^{\frac{p}{2}} = \ell^p(t) \text{ for } \alpha > 1 \quad p < 2\alpha - 1 \end{cases}$$

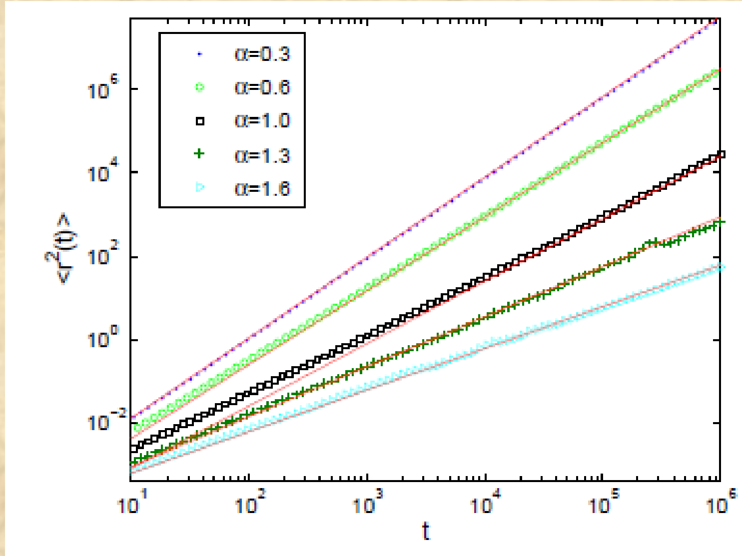
Anomalous diffusion (first case)

Anomalous diffusion with Strongly Anomalous exponents (second and third cases)

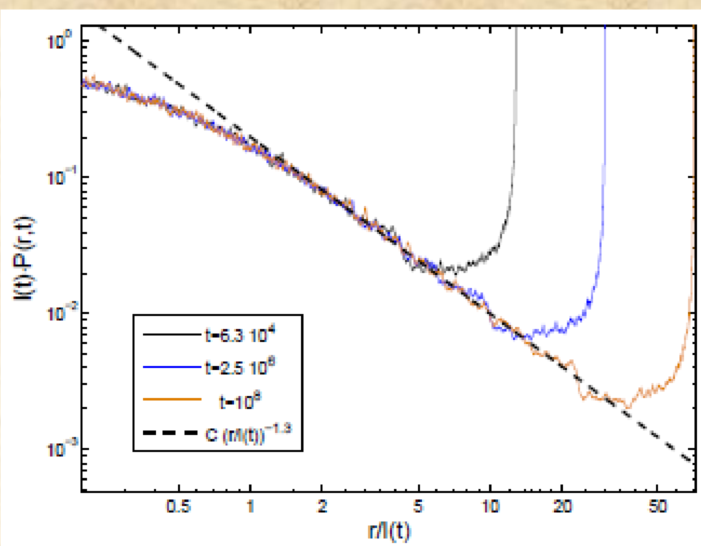
Normal diffusion (fourth case)

NUMERICAL RESULTS

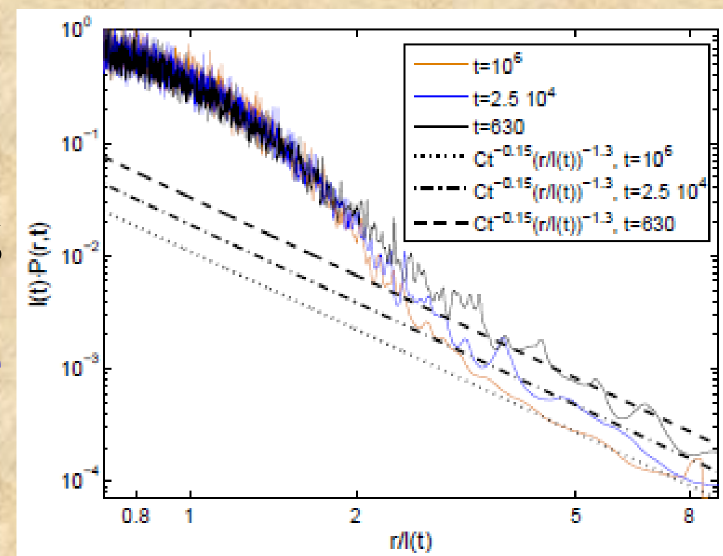
Analythic approximated results compared with numeric simulations, i.e. Montecarlo smulations



Mean square displacement as a function of time compared with theory for different *α*'s



Very good agreement between theory and simulations, scaling and single jump hypothesis could be exact



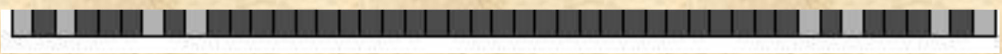
Long tail of *P*(*r*,*t*) for *α*=0.3 case b)
dashed line represent theory theory

Long tail of *P*(*r*,*t*) vanishing with *t* for *α*=1.3 case a)

LEVY QUASICRYSTAL

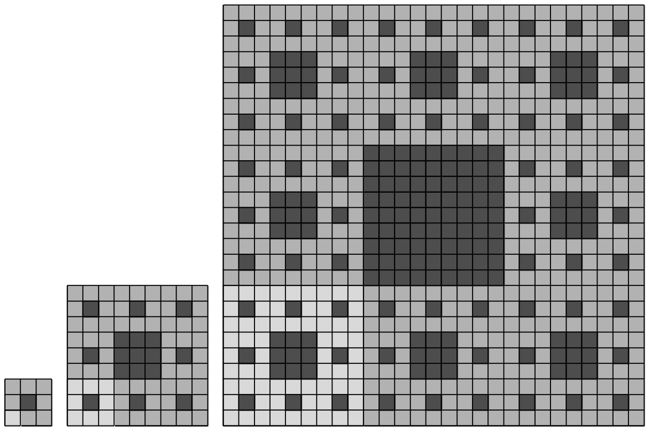
1, 2 and 3 (*d*) dimensional Lèvy Quasicrystal are deterministic fractals where holes (dark squares) are distributed according to a Levy distribution.

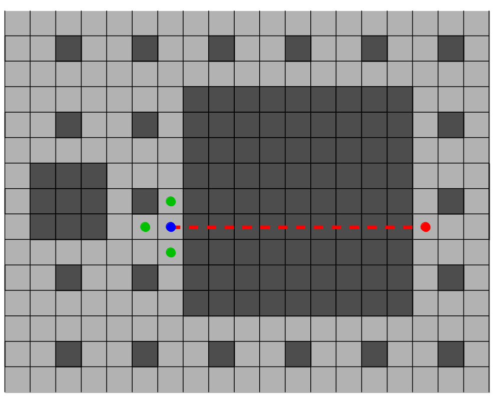
1-dimensional Cantor Fractal



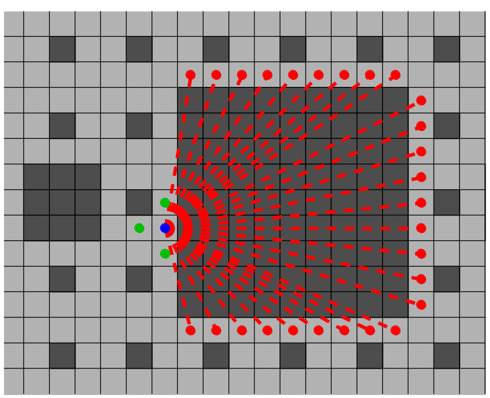
In 2-3 dimension different jumping rules. Difference depends on large scale features of the jumps. Results depends on the jump rule.

2-dimensional Sierpinski Carpet





Straight jumps.
Head on dynamics



Random diagonal jumps.
Fan out dynamics

SAME SCALING PICTURE

$$\ell(t) \approx t^{1/z}$$

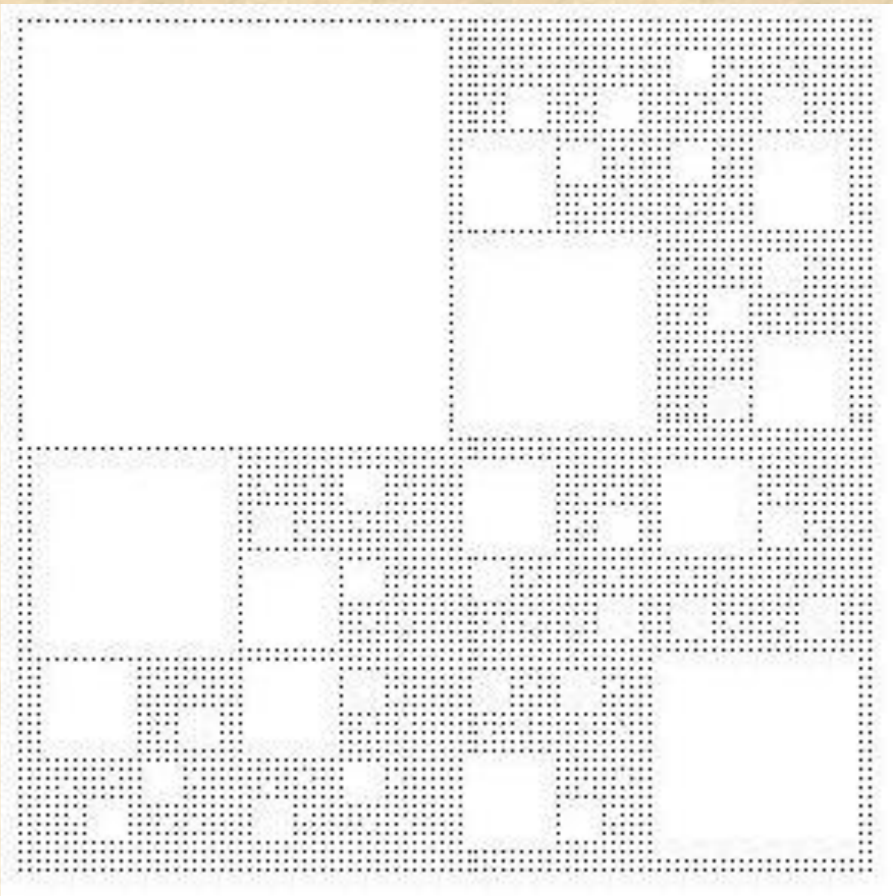
-Value of *z* analytically known only in 1d systems

-Numerical results: z different of the annealed result.

- Loacal vs. Average

-Single long jumps provides strong anomalous diffusion

PERSPECTIVES



-Random case in 2-3 dimension, the construction (definition) of the structures is a non trivial problem. Packing.

Differences seem to be present with respect the annealed case.
arXiv:1105.4149 C. W. Groth,
A. R. Akhmerov and C. W. J. Beenakker

Randomization procedure seems to be a relevant parameter.

- Is there a upper critical dimension where annealed and quenched model are the same at least for random system?

- Rigorous proof (without scaling hypothesis) for results in one-dimension

- **R. Burioni, L. Caniparoli, A. Vezzani PRE E 81, 060101(R) (2010)**
- **R. Burioni, L. Caniparoli, S. Lepri and A. Vezzani PRE 81, 011127 (2010)**
- **A. Vezzani, R. Burioni, L. Caniparoli, and S. Lepri, Phil.Mag. 91, 1987 (2011)**
- **P. Buonsante, R. Burioni, and A. Vezzani PRE 84, 021105 (2011)**