Random Matrix Theory with Power-Law Tails

SMFT Bari 04/09/2008

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Outline:

1. **Motivation:** Applications
2. **What is Random Matrix Theory (RMT)?**
   – basic properties and results
3. **Why Power Laws:**
   – comparison to data
   – new universality classes
4. **Open Problems**

Motivation: RMT and Applications

- **general idea:**

  describe complicated = many body, interacting system
  by random variables

$$H\Psi_i = \lambda_i \Psi_i \quad \longrightarrow \quad \exp[-\sum H_{ij} H^\dagger_{ji}]$$

  e.g.: Hamiltonian of Nucleus or Quantum Billiard,
  Dirac Operator $\not{D}$ in **QCD**

- [Bohigas, Giannoni, Schmidt / Casati, Vals-Gris, Guarnieri 80's]:

  Quantum Chaotic Systems display RMT Statistics!

- **RMT: eigenvalues correlated**

  – based on global symmetries: universal

  – analytically solvable for: spectrum $\rho(\lambda) \sim \langle\, \sum_i \delta(\lambda - \lambda_i)\,\rangle$,
  individual eigenvalues, consecutive eigenvalue spacings

What is RMT

- $N \times T$ **matrix:** $\mathbf{X}_{ij} \in \mathbb{R}/\mathbb{C}/\mathbb{H}$
  where $i = 1, \ldots, N$ and $j = 1, \ldots, T$

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  $V(\lambda) = n\lambda$ Gauß: 1 parameter
  Wishard/Laguerre/chiral Ensemble

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- **correlation functions:** $\boxed{\langle\ \mathbb{O}\ \rangle \equiv \mathcal{Z}^{-1} \int d\mathbf{X}\ \mathbb{O}(\mathbf{X})\ P[\mathbf{X}]}$
  e.g. density $\rho(\lambda) = \langle\ \mathrm{Tr}\delta(\lambda - \mathbf{X}\mathbf{X}^\dagger)\ \rangle$

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- **eigenvalues:** $\mathbf{X}\mathbf{X}^\dagger \to U \Lambda U^\dagger$ with $\lambda_{i=1,\ldots,N} \in \mathbb{R}_+$

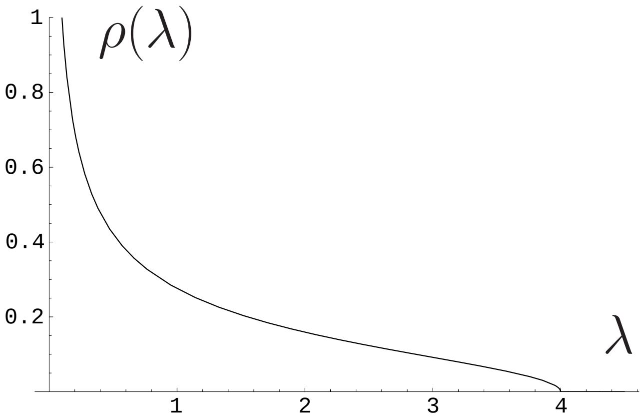
$$\boxed{\mathcal{Z} = \int d\mathbf{X} P[\mathbf{X}] \to \int_0^\infty \prod_i d\lambda_i\ \lambda_i^{T-N} e^{-NV(\lambda_i)} \prod_{k>l} |\lambda_k - \lambda_j|^2}$$

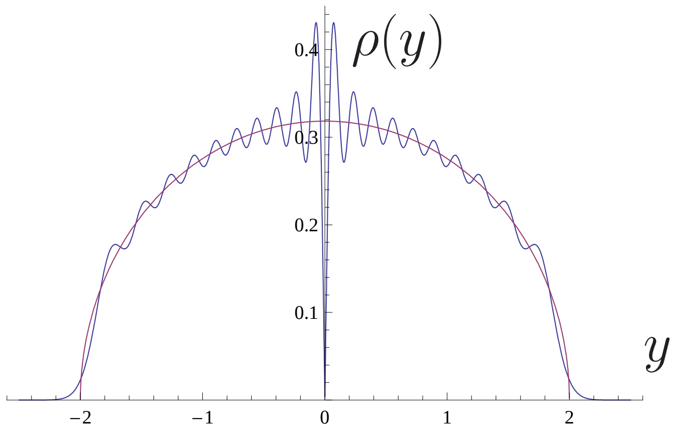
$\to$ **2D Coulomb gas on 1D:**

$$\boxed{\mathcal{S} = (T - N) \sum_i \log[\lambda_i] - N \sum_i V(\lambda_i) + \sum_{k \neq l} \log|\lambda_k - \lambda_l|}$$

Large-N Limits

1) semi-circle: $T - N = \mathcal{O}(1)$



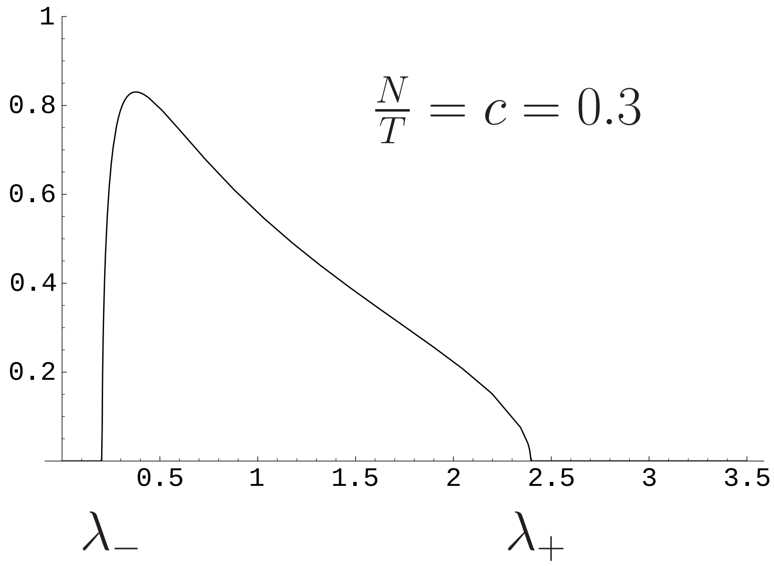


$\rho(\lambda) \sim \sqrt{\frac{4}{\lambda} - 1}$ or in variable $\lambda \longrightarrow y^2$: $\hat{\rho}(y) \sim \sqrt{4 - \lambda^2}$

- **local behaviour universal:** zoom $x = (\lambda - \lambda_0) N^{\delta}$

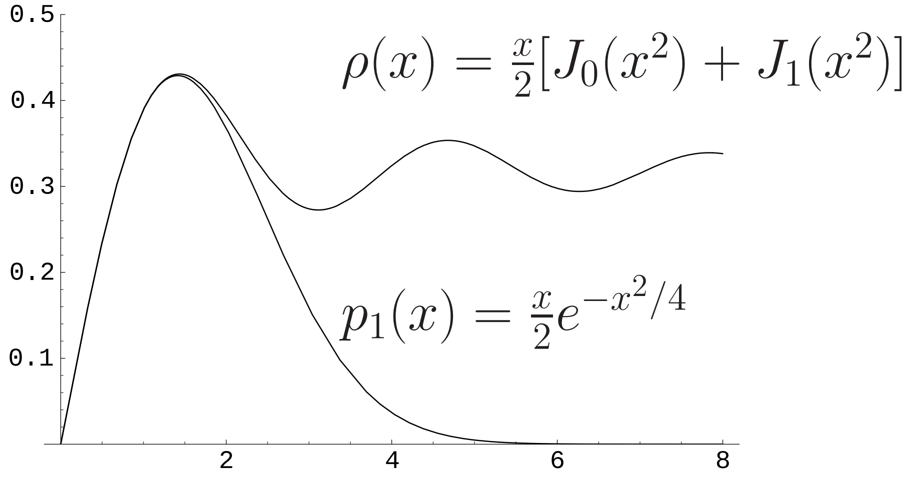
correlations: orgin = Bessel; bulk = Sine; edge = Airy

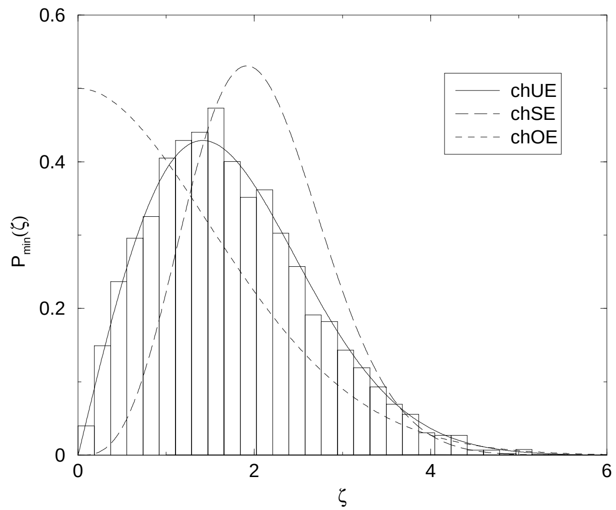
2) Marčenko-Pastur: $T - N = \mathcal{O}(N)$

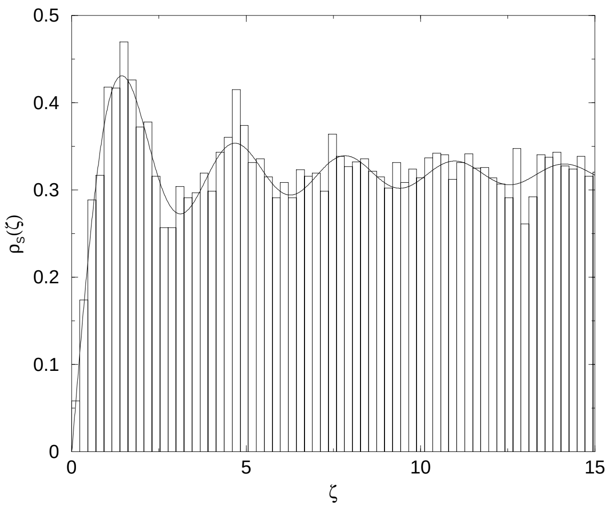


$$\rho(\lambda) \sim \frac{1}{\lambda}\sqrt{(\lambda - \lambda_+(c))(\lambda_-(c) - \lambda)}$$

Example 1): Origin Limit vs. QCD

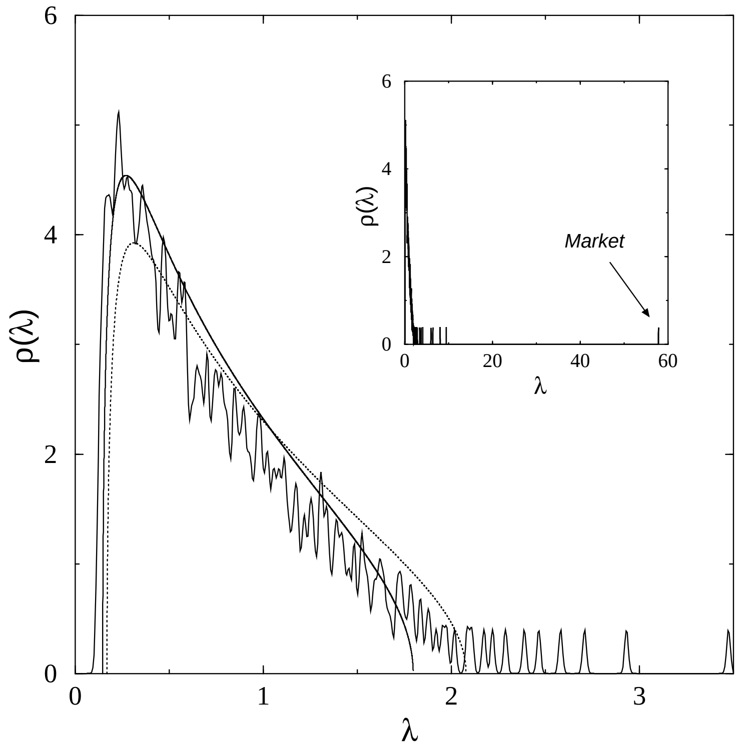






[Damgaard et al., 98, 01] $x = \lambda_{\not{D}} V \langle \bar{q}q \rangle$ rescaled ev. $\not{D} = \begin{pmatrix} 0 & i\mathbf{X} \\ i\mathbf{X}^\dagger & 0 \end{pmatrix}$

Example 2): M-P vs. Finance



[Plerou et al., 01]

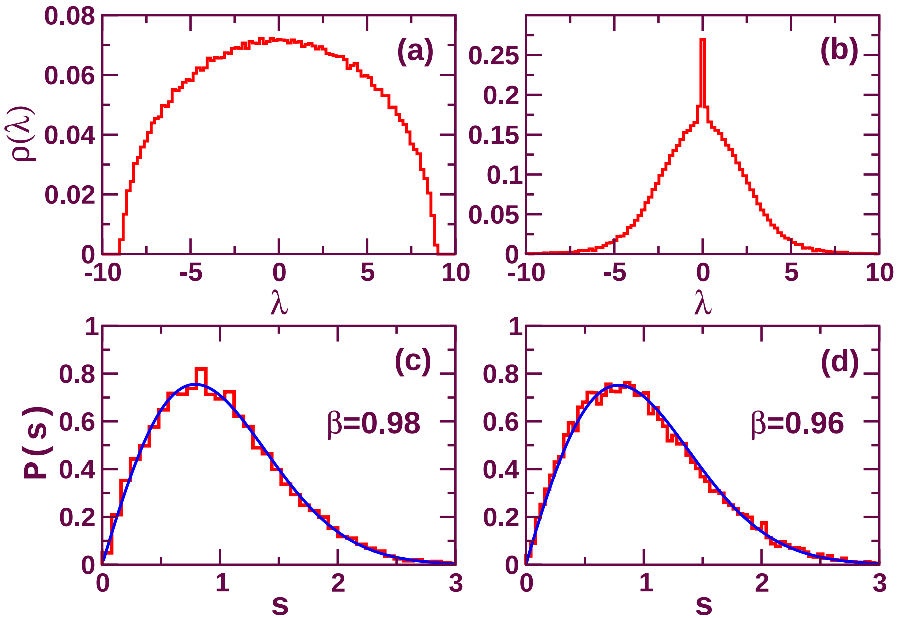
Correlation matrices of times series of stock prices $S_i(t)$

$$X_{it} = \sigma_i{}^{-1}[G_i(t) - \langle G_i(t) \rangle]$$

with normalised return $G_i(t) = \log[S_i(t+\Delta t)] - \log[S_i(t)]$

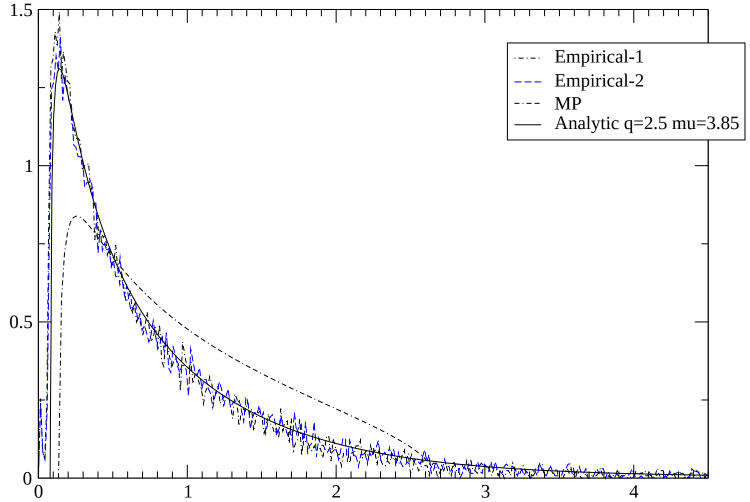
Where are the Power-Laws?

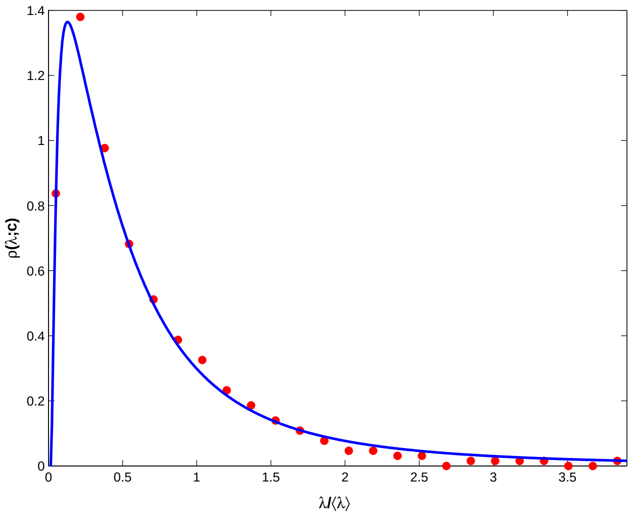
- general feature in complex networks: [Barabási, Albert 99; Bandyopadhyay, Jalan 07]



spectrum of **adjacency matrix** of random network (Erdös-Reyni) **left** and scale-free network **right**

• stock prices have power law tails [Biroli, Bouchaud, Potters 07; G.A., Fischmann, Vivo 08]





RMTs with Heavy Tails

- **correlation matrix** $\langle \mathbf{X}_{it}\mathbf{X}_{jt'}\rangle \sim \mathbf{C}_{ij}\mathbf{A}_{tt'}$

- **empirical correlation matrix** $c_{ij} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{X}_{it}\mathbf{X}_{jt}$

  – compare to Gauß$\frac{1}{T}\mathrm{Tr}\mathbf{X}\mathbf{X}^{\dagger}$: Marčenko-Pastur

- **Power-Law RMT: non-invariant**

  – incl. correlation $\mathrm{Tr}\mathbf{X}\mathbf{X}^{\dagger} \to \mathrm{Tr}\mathbf{X}\mathbf{C}^{-1}\mathbf{X}^{\dagger}\mathbf{A}^{-1}$ [Burda et al. 07]

  – variance random variable: $\mathbf{X}_{it} \to \sigma^{t}\mathbf{X}_{it}$ [Biroli et al. 07]

- **Power-Law RMT: invariant** [Bohigas et al. 04; G.A., Vivo 08]

$$P[\mathbf{X}\mathbf{X}^{\dagger}] \to \frac{1}{(1+\gamma^{-1}\mathrm{Tr}\mathbf{X}\mathbf{X}^{\dagger})^{\gamma}}$$

  – can be derived from generalised entropy principle

Analytic Results

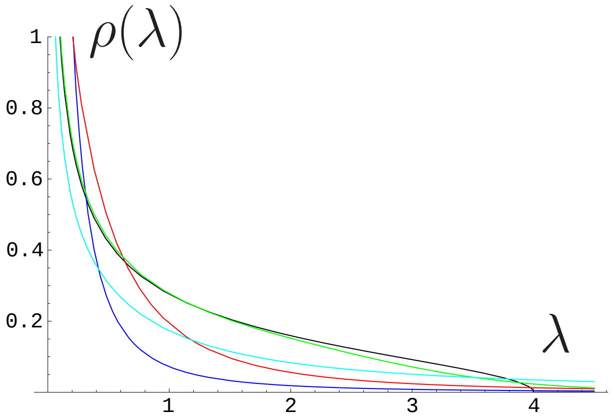
- **fold** know formulae for **Gauß**

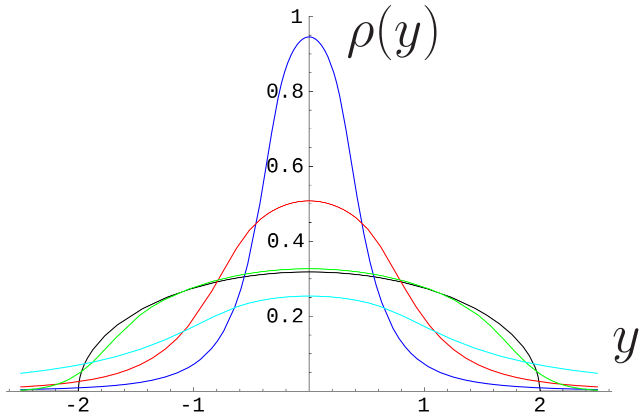
$$P[\mathbf{X}\mathbf{X}^\dagger] \sim \int d\xi\; e^{-\xi}\xi^{\gamma-1}\exp\left[-\xi\frac{1}{\gamma}\mathrm{Tr}\mathbf{X}\mathbf{X}^\dagger\right]$$

- Result: **one parameter deformation** (fixed $c = \frac{N}{T}$)

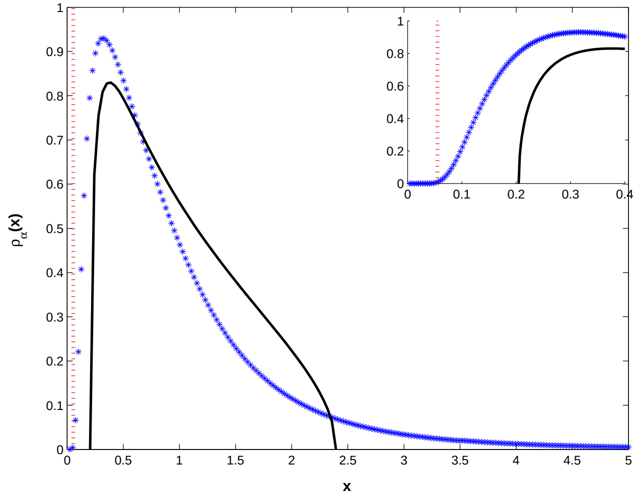
$\alpha \sim \gamma - NT$ $\boxed{\rho(\lambda) \sim \lambda^{-\alpha}, \quad 1 < \alpha \neq 2}$

- example $\int$ semi-circle:



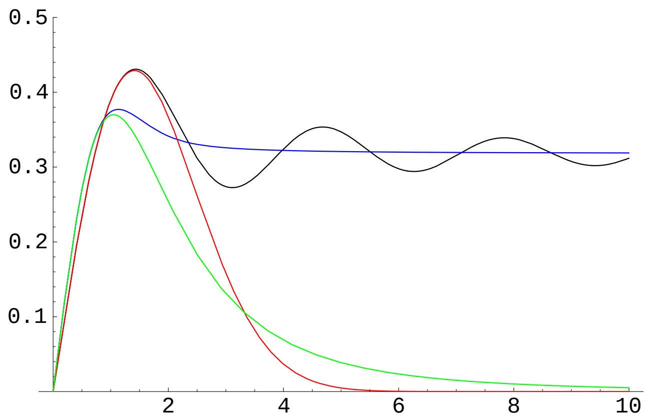


- example $\int$ Marčenko-Pastur:



$$\frac{N}{T} = c = 0.3, \;\; \alpha = 5$$

- example $\int$ origin=Bessel: **universal**



$$N = T, \;\; \alpha = 2.1$$

– even smallest eigenvalue has a tail!

Open Problems

- in ordinary RMT: largest ev λ_{max} has many applications, what happens with power-law?

- distribution of smallest eigenvalue for $c < 1$?

- (some) universality:
  similar curves from multivariate student distribution

- comparison to other data, e.g. from complex networks

- how to subtract noise = RMT from data?